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NORMAL MATRIX COMPRESSIONS

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Abstract. There has been longstanding interest in the problem of characterizing normal compressions of normal matrices. Indeed, the solution to the Hermitian case goes back to the Cauchy interlacing theorem, and its converse (due to Fan and Pall). More recently, the theory of higher–rank numerical ranges has included the solution in the case of scalar compressions. Here we take steps towards a similar treatment of the general case. We develop some natural necessary conditions on the eigenvalues as well as some convenient sufficient conditions, showing by a study of the 2x2 compressions of 4x4 normals that the necessary conditions are not sufficient. We also give a new proof of the Choi–Kribs–Życzkowski conjecture for 2x2 compressions by means of a powerful extension of that result. The CKŻ conjecture (more recently a theorem) may be stated as follows: given an $N\times N$ normal matrix M with eigenvalues $\lambda_1,\dots,\lambda_N$, the set of $a\in\mathbb{C}$ for which the scalar matrix aI_k is a compression of M is precisely

$$\Omega_k(M)=\bigcap_{\#(J)=N-k+1}\operatorname{conv}\{\lambda_j : j\in J\}.$$

Thus, for $k=2$ we see that $a\in\Omega_2(M)$ implies that $\operatorname{diag}(a,a)$ is a compression of M (the reverse implication is relatively straightforward). We show that, in fact, for *any pair* $a,b\in\Omega_2(M)$, $\operatorname{diag}(a,b)$ is a compression of M. Our proof is independent of the earlier results and depends on a novel approach. We also study the continuity of the map $a\to B(a)$, where $B(a)$ denotes the set of all $b\in\mathbb{C}$ such that $\operatorname{diag}(a,b)$ is a compression of M.

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