BOUNDARY DATA MAPS AND KREIN'S RESOLVENT FORMULA FOR STURM—LIOUVILLE OPERATORS ON A FINITE INTERVAL

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Abstract. We continue the study of boundary data maps, that is, generalizations of spectral parameter dependent Dirichlet-to-Neumann maps for (three-coefficient) Sturm–Liouville operators on the finite interval (a, b), to more general boundary conditions, began in [8] and [17]. While these earlier studies of boundary data maps focused on the case of general separated boundary conditions at a and b, the present work develops a unified treatment for all possible self-adjoint boundary conditions (i.e., separated as well as non-separated ones).

In the course of this paper we describe the connections with Krein's resolvent formula for self-adjoint extensions of the underlying minimal Sturm–Liouville operator (parametrized in terms of boundary conditions), with some emphasis on the Krein extension, develop the basic trace formulas for resolvent differences of self-adjoint extensions, especially, in terms of the associated spectral shift functions, and describe the connections between various parametrizations of all self-adjoint extensions, including the precise relation to von Neumann's basic parametrization in terms of unitary maps between deficiency subspaces.

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