

## EXISTENCE OF MAXIMAL SEMIDEFINITE INVARIANT SUBSPACES AND SEMIGROUP PROPERTIES OF SOME CLASSES OF ORDINARY DIFFERENTIAL OPERATORS

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*Abstract.* We describe sufficient conditions for the operator  $Lu = \frac{1}{g(x)}L_0u$ , with  $L_0$  an ordinary differential operator dissipative on its domain and a function  $g$  changing its sign, to have maximal semidefinite invariant subspaces in the Krein space  $L_{2,g}(a,b)$  with the indefinite inner product  $[u,v] = \int_a^b g(x)u(x)\overline{v(x)}dx$ . The semigroup properties of the restrictions of an operator to these subspaces are studied. The similarity problem of  $L$  to a selfadjoint operator is discussed.

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