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PRE–IMAGES OF BOUNDARY POINTS OF THE NUMERICAL RANGE

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Abstract. This paper considers matrices $A \in M_n(\mathbb{C})$ whose numerical range contains boundary points generated by multiple linearly independent vectors. Sharp bounds for the maximum number of such boundary points (excluding flat portions) are given for unitarily irreducible matrices of dimension $\leqslant 5$. An example is provided to show that there may be infinitely many for $n = 6$. For matrices unitarily similar to tridiagonal, however, a finite upper bound is found for all n. A somewhat unexpected byproduct of this is an explicit example of $A \in M_5(\mathbb{C})$ which is not tridiagonalizable via a unitary similarity.

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