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THE BLOCK NUMERICAL RANGE OF ANALYTIC OPERATOR FUNCTIONS

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Abstract. We introduce the block numerical range $W^n(\mathscr{L})$ of an operator function $\mathscr{L}$ with respect to a decomposition $H = H_1 \oplus \ldots \oplus H_n$ of the underlying Hilbert space. Our main results include the spectral inclusion property and estimates of the norm of the resolvent for analytic $\mathscr{L}$. They generalise, and improve, the corresponding results for the numerical range (which is the case $n = 1$) since the block numerical range is contained in, and may be much smaller than, the usual numerical range. We show that refinements of the decomposition entail inclusions between the corresponding block numerical ranges and that the block numerical range of the operator matrix function $\mathscr{L}$ contains those of its principal subminors. For the special case of operator polynomials, we investigate the boundedness of $W^n(\mathscr{L})$ and we prove a Perron-Frobenius type result for the block numerical radius of monic operator polynomials with coefficients that are positive in Hilbert lattice sense.

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