

A SURVEY ON THE BÖTTCHER–WENZEL CONJECTURE AND RELATED PROBLEMS

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Abstract. A fundamental fact in matrix theory is that the matrix multiplication is not commutative, i.e., there are square matrices X and Y such that $XY \neq YX$. The difference $XY - YX$ is called the commutator (or Lie product) of X and Y . The commutator plays an important role in diverse areas in mathematics, for instance, Lie group and Lie algebra theory, perturbation analysis, and matrix manifold computation. Böttcher and Wenzel proposed the following conjecture in 2005 : for any real $n \times n$ matrices X and Y ,

$$\|XY - YX\|_F \leq \sqrt{2} \|X\|_F \|Y\|_F,$$

where $\|\cdot\|_F$ is the Frobenius norm. This survey is concerned with the proofs of this conjecture and the study of its related problems.

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