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CRAWFORD NUMBERS OF COMPANION MATRICES

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Abstract. The (generalized) Crawford number $C(A)$ of an n-by-n complex matrix A is, by definition, the distance from the origin to the boundary of the numerical range $W(A)$ of A. If A is a companion matrix

$$\begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & 0 & 1 \\ -a_n & -a_{n-1} & \cdots & \cdots & -a_2 & -a_1 \end{bmatrix},$$

then it is easily seen that $C(A) \geqslant \cos(\pi/n)$. The main purpose of this paper is to determine when the equality $C(A) = \cos(\pi/n)$ holds. A sufficient condition for this is that the boundary of $W(A)$ contains a point λ for which the subspace of $\mathbb{C}^n$ spanned by the vectors x with $\langle Ax,x\rangle = \lambda\|x\|^2$ has dimension 2, while a necessary condition is $\sum_{j=0}^{n-2} a_{n-j}e^{(n-j)i\theta}\sin\big((j+1)\pi/n\big) = \sin(\pi/n)$ for some real θ. Examples are given showing that in general these conditions are not simultaneously necessary and sufficient. We then prove that they are if A is (unitarily) reducible. We also establish a lower bound for the numerical radius $w(A)$ of A: $w(A) \geqslant \cos(\pi/(n+1))$, and show that the equality holds if and only if A is equal to the n-by-n Jordan block.

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REFERENCES

[1] C.-C. CHANG, H.-L. GAU, Y.-S. WANG, S.-C. WU AND Y.-T. YEH, *Matrices with defect index one*, Oper. Matrices **7** (2013), 865–885.
[2] S. H. CHENG AND N. J. HIGHAM, *The nearest definite pair for the Hermitian generalized eigenvalue problem*, Linear Algebra Appl. **302/303** (1999), 63–76.
[3] C. R. CRAWFORD, *A stable generalized eigenvalue problem*, SIAM J. Numer. Anal. **13** (1976), 854–860; *Errata*, SIAM J. Numer. Anal. **15** (1978), 1070.
[4] M. R. EMBRY, *The numerical range of an operator*, Pacific J. Math. **32** (1970), 647–650.
[5] H.-L. GAU, *Numerical ranges of reducible companion matrices*, Linear Algebra Appl. **432** (2010), 1310–1321.
[6] H.-L. GAU AND P. Y. WU, *Numerical range of $S(\phi)$*, Linear Multilinear Algebra **45** (1998), 49–73.
[7] H.-L. GAU AND P. Y. WU, *Companion matrices: reducibility, numerical ranges and similarity to contractions*, Linear Algebra Appl. **383** (2004), 127–142.
[8] H.-L. GAU AND P. Y. WU, *Numerical ranges of companion matrices*, Linear Algebra Appl. **421** (2007), 202–218.
[9] U. HAAGERUP AND P. DE LA HARPE, *The numerical radius of a nilpotent operator on a Hilbert space*, Proc. Amer. Math. Soc. **115** (1992), 371–379.
[10] R. A. HORN AND C. R. JOHNSON, *Matrix Analysis*, 2nd ed., Cambridge University Press, Cambridge, 2013.
[11] R. A. HORN AND C. R. JOHNSON, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991.

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[12] M. Marcus and B. N. Shure, *The numerical range of certain* $0,1$ *-matrices*, Linear Multilinear Algebra **7** (1979), 111–120.
[13] G. Polya and G. Szegő, *Problems and Theorems in Analysis*, Vol. II, Springer, Berlin, 1976.
[14] G. W. Stewart, *Perturbation bounds for the definite generalized eigenvalue problem*, Linear Algebra Appl. **23** (1979), 69–85.
[15] K.-Z. Wang, P. Y. Wu and H.-L. Gau, *Crawford numbers of powers of a matrix*, Linear Algebra Appl. **433** (2010), 2243–2254.