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ON THE INDEX OF A NON–FREDHOLM MODEL OPERATOR

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Abstract. Let $\{A(t)\}_{t\in\mathbb{R}}$ be a path of self-adjoint Fredholm operators in a Hilbert space $\mathscr{H}$, joining endpoints $A_\pm$ as $t\to\pm\infty$. Computing the index of the operator $\boldsymbol{D_A} = \partial/\partial t + \boldsymbol{A}$ acting on $L^2(\mathbb{R};\mathscr{H})$, where $\boldsymbol{A}$ denotes the multiplication operator $(\boldsymbol{A}f)(t) = A(t)f(t)$ for $f\in L^2(\mathbb{R};\mathscr{H})$, and its relation to spectral flow along this path, has a long history, but it is mostly focussed on the case where the operators $A(t)$ all have purely discrete spectrum.

Introducing the operators $\boldsymbol{H}_1 = \boldsymbol{D_A}^*\boldsymbol{D_A}$ and $\boldsymbol{H}_2 = \boldsymbol{D_A}\boldsymbol{D_A}^*$, we consider spectral shift functions, denoted by $\xi(\cdot;A_+,A_-)$ and $\xi(\cdot;\boldsymbol{H}_2,\boldsymbol{H}_1)$ associated with the pairs (A_+,A_-) and $(\boldsymbol{H}_2,\boldsymbol{H}_1)$. Under the restrictive hypotheses that A_+ is a relatively trace class perturbation of A_-, a relationship between these spectral shift functions was proved in [14], for certain operators $A_\pm$ with essential spectrum, extending a result of Pushnitski [22]. Moreover, assuming $A_\pm$ to be Fredholm, the value $\xi(0;A_-,A_+)$ was shown to represent the spectral flow along the path $\{A(t)\}_{t\in\mathbb{R}}$ while that of $\xi(0_+;\boldsymbol{H}_1,\boldsymbol{H}_2)$ yields the Fredholm index of $\boldsymbol{D_A}$. The fact, proved in [14], that these values of the two spectral functions are equal, resolves the index = spectral flow question in this case. This relationship between spectral shift functions was generalized to non-Fredholm operators in [9] again under the relatively trace class perturbation hypothesis. In this situation it asserts that the Witten index of $\boldsymbol{D_A}$, denoted by $W_r(\boldsymbol{D_A})$, a substitute for the Fredholm index in the absence of the Fredholm property of $\boldsymbol{D_A}$, is given by

$$W_r(\boldsymbol{D_A}) = \xi_L(0_+;\boldsymbol{H}_2,\boldsymbol{H}_1) = [\xi_L(0_+;A_+,A_-)+\xi_L(0_-;A_+,A_-)]/2.$$

Here one assumes that $\xi(\cdot;A_-,A_+)$ possesses a right and left Lebesgue point at 0 denoted by $\xi_L(0_\pm;A_+,A_-)$ (and similarly for $\xi_L(0_+;\boldsymbol{H}_2,\boldsymbol{H}_1)$).

When the path $\{A(t)\}_{t\in\mathbb{R}}$ consists of differential operators, the relatively trace class perturbation assumption is violated. The simplest assumption that applies (to differential operators in 1+1 dimensions) is to admit relatively Hilbert–Schmidt perturbations. This is not just an incremental improvement. In fact, the method we employ here to make this extension is of interest in any dimension. Moreover we consider $A_\pm$ which are not necessarily Fredholm and we establish that the relationships between the two spectral shift functions found in **all** of the previous papers [9] ,[14], and [22], can be proved, even in the non-Fredholm case. The significance of our new methods is that, besides being simpler, they also allow a wide class of examples such as pseudodifferential operators in higher dimensions. Most importantly, we prove the above formula for the Witten index in the most general circumstances to date.

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