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SOLUTIONS OF A CONSTRAINED HERMITIAN MATRIX–VALUED FUNCTION OPTIMIZATION PROBLEM WITH APPLICATIONS

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Abstract. Let $f(X)=(XC+D)M(XC+D)^*-G$ be a given nonlinear Hermitian matrix-valued function with $M=M^*$ and $G=G^*$, and assume that the variable matrix X satisfies the consistent linear matrix equation $XA=B$. This paper shows how to characterize the semi-definiteness of $f(X)$ subject to all solutions of $XA=B$. As applications, a standard method is obtained for finding analytical solutions X_0 of $X_0A=B$ such that the matrix inequality $f(X)\succcurlyeq f(X_0)$ or $f(X)\preccurlyeq f(X_0)$ holds for all solutions of $XA=B$. The whole work provides direct access, as a standard example, to a very simple algebraic treatment of the constrained Hermitian matrix-valued function and the corresponding semi-definiteness and optimization problems.

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