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ON THE NORMALIZED NUMERICAL RANGE

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Abstract. The normalized numerical range of an operator A is defined as the set $F_N(A)$ of all the values $\langle Ax,x\rangle / \|Ax\|$ attained by unit vectors $x \notin \ker A$. We prove that $F_N(A)$ is simply connected, establish conditions for it to be star-shaped with the center at zero, to be open, closed, and to have empty interior. For some classes of operators (weighted shifts, isometries, essentially Hermitian) the complete description of $F_N(A)$ is obtained.

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REFERENCES

[1] W. AUZINGER, *Sectorial operators and normalized numerical range*, Appl. Numer. Math. **45** (4): 367–388, 2003.
[2] D. COREY, C. JOHNSON, R. KIRK, B. LINS, AND I. M. SPITKOVSKY, *The product field of values*, Linear Algebra Appl. **438**: 2155–2173, 2013.
[3] JU. L. DALECKII AND M. G. KREIN, *Stability of solutions of differential equations in Banach space*, American Mathematical Society, Providence, R. I., 1974.
[4] D. S. DJORDJEVIĆ, *Characterizations of normal, hyponormal and EP operators*, J. Math. Anal. Appl. **329** (2): 1181–1190, 2007.
[5] E. DURSZT, *On the numerical range of normal operators*, Acta Sci. Math. (Szeged), **25**: 262–265, 1964.
[6] L. Z. GEVORGYAN, *On the convergence rate of iterations and the normalized numerical range of an operator*, Math. Sci. Res. J. **8** (1): 16–26, 2004.
[7] L. Z. GEVORGYAN, *On some properties of the normalized numerical range*, Izv. Nats. Akad. Nauk Armenii Mat. **41** (1): 41–48, 2006.
[8] L. Z. GEVORGYAN, *An example of the normalized numerical range*, Armenian J. Math. **1** (1): 50–33, 2009.
[9] L. Z. GEVORGYAN, *Normalized numerical ranges of some operators*, Operators and Matrices **3** (1): 145–153, 2009.
[10] L. Z. GEVORGYAN, *Normalized numerical ranges of some complex* 2×2 *matrices*, Izv. Nats. Akad. Nauk Armenii Mat. **46** (5): 41–52, 2011.
[11] K. E. GUSTAFSON AND D. K. M. RAO, *Numerical Range. The Field of Values of Linear Operators and Matrices*, Springer, New York, 1997.
[12] P. HALMOS, *A Hilbert Space Problem Book*, Van Nostrand, Princeton, N. J., 1967.
[13] R. A. HORN AND C. R. JOHNSON, *Topics in matrix analysis*, Cambridge University Press, Cambridge, 1994. Corrected reprint of the 1991 original.
[14] L. V. KANTOROVICH, *Functional analysis and applied mathematics*, Uspehi Matem. Nauk (N. S.), **3** (6 (28)): 89–185, 1948.
[15] L. V. KANTOROVICH, *Functional analysis and applied mathematics*, NBS Rep. 1509. U. S. Department of Commerce, National Bureau of Standards, Los Angeles, Calif., 1952., translated by C. D. Benster.
[16] W. C. RIDGE, *Numerical range of a weighted shift with periodic weights*, Proc. Amer. Math. Soc. **55**: 107–110, 1976.
[17] Q. F. STOUT, *The numerical range of a weighted shift*, Proc. Amer. Math. Soc. **88**: 495–502, 1983.

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[18] B. SZ.-NAGY, C. FOIAS, H. BERCOVICI, AND L. KÉRCHY, *Harmonic analysis of operators on Hilbert space*, Universitext, Springer, New York, second edition, 2010.
[19] K.-Z. WANG AND P. Y. WU, *Numerical ranges of weighted shifts*, J. Math. Anal. Appl. **381** (2): 897–909, 2011.