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PSEUDOSPECTRUM OF AN ELEMENT OF A BANACH ALGEBRA

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Abstract. The ε-pseudospectrum $\Lambda_\varepsilon(a)$ of an element a of an arbitrary Banach algebra A is studied. Its relationships with the spectrum and numerical range of a are given. Characterizations of scalar, Hermitian and Hermitian idempotent elements by means of their pseudospectra are given. The stability of the pseudospectrum is discussed. It is shown that the pseudospectrum has no isolated points, and has a finite number of components, each containing an element of the spectrum of a. Suppose for some $\varepsilon > 0$ and $a, b \in A$, $\Lambda_\varepsilon(ax) = \Lambda_\varepsilon(bx)$ $\forall x \in A$. It is shown that $a = b$ if:

(i) a is invertible.

(ii) a is Hermitian idempotent.

(iii) a is the product of a Hermitian idempotent and an invertible element.

(iv) A is semisimple and a is the product of an idempotent and an invertible element.

(v) $A = B(X)$ for a Banach space X.

(vi) A is a C^*-algebra.

(vii) A is a commutative semisimple Banach algebra.

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