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SPECTRAL ANALYSIS OF NON–SELF–ADJOINT JACOBI OPERATOR ASSOCIATED WITH JACOBIAN ELLIPTIC FUNCTIONS

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Abstract. We perform the spectral analysis of a family of Jacobi operators $J(\alpha)$ depending on a complex parameter α. If $|\alpha| \neq 1$ the spectrum of $J(\alpha)$ is discrete and formulas for eigenvalues and eigenvectors are established in terms of elliptic integrals and Jacobian elliptic functions. If $|\alpha| = 1$, $\alpha \neq \pm 1$, the essential spectrum of $J(\alpha)$ covers the entire complex plane. In addition, a formula for the Weyl m-function as well as the asymptotic expansions of solutions of the difference equation corresponding to $J(\alpha)$ are obtained. Finally, the completeness of eigenvectors and Rodriguez-like formulas for orthogonal polynomials, studied previously by Carlitz, are proved.

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