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MINIMAL SCALINGS AND STRUCTURAL PROPERTIES OF SCALABLE FRAMES

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Abstract. For a unit-norm frame $F = \{f_i\}_{i=1}^k$ in $\mathbb{R}^n$, a scaling is a vector $c = (c(1),\dots,c(k)) \in \mathbb{R}^k_{\geqslant 0}$ such that $\{\sqrt{c(i)}f_i\}_{i=1}^k$ is a Parseval frame in $\mathbb{R}^n$. If such a scaling exists, F is said to be scalable. A scaling c is a minimal scaling if $\{f_i : c(i) > 0\}$ has no proper scalable subframe. In this paper, we provide an algorithm to find all possible contact points for the John's decomposition of the identity by applying the b-rule algorithm to a linear system which is associated with a scalable frame. We also give an estimate of the number of minimal scalings of a scalable frame. We provide a characterization of when minimal scalings are affinely dependent. Using this characterization, we can conclude that all strict scalings $c = (c(1),\dots,c(k)) \in \mathbb{R}^k_{>0}$ of F have the same structural property. That is, the collections of all tight subframes of strictly scaled frames are the same up to a permutation of the frame elements. We also present the uniqueness of orthogonal partitioning property of any set of minimal scalings, which provides all possible tight subframes of a given scaled frame.

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Paper OaM-11-73

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