

RESONANCE FREE REGIONS AND NON-HERMITIAN SPECTRAL OPTIMIZATION FOR SCHRÖDINGER POINT INTERACTIONS

SERGIO ALBEVERIO AND ILLYA M. KARABASH

Abstract. Resonances of Schrödinger Hamiltonians with point interactions are considered. The main object under the study is the resonance free region under the assumption that the centers, where the point interactions are located, are known and the associated “strength” parameters are unknown and allowed to bear additional dissipative effects. To this end we consider the boundary of the resonance free region as a Pareto optimal frontier and study the corresponding optimization problem for resonances. It is shown that upper logarithmic bound on resonances can be made uniform with respect to the strength parameters. The necessary conditions on optimality are obtained in terms of first principal minors of the characteristic determinant. We demonstrate the applicability of these optimality conditions on the case of 4 equidistant centers by computing explicitly the resonances of minimal decay for all frequencies. This example shows that a resonance of minimal decay is not necessarily simple, and in some cases it is generated by an infinite family of feasible resonators.

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