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COFACTORS AND EIGENVECTORS OF BANDED TOEPLITZ MATRICES: TRENCH FORMULAS VIA SKEW SCHUR POLYNOMIALS

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Abstract. The Jacobi–Trudi formulas imply that the minors of the banded Toeplitz matrices can be written as certain skew Schur polynomials. In 2012, Alexandersson expressed the corresponding skew partitions in terms of the indices of the struck-out rows and columns. In the present paper, we develop the same idea and obtain some new applications. First, we prove a slight generalization and modification of Alexandersson's formula. Then, we deduce corollaries about the cofactors and eigenvectors of banded Toeplitz matrices, and give new simple proofs to the corresponding formulas published by Trench in 1985.

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