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EIGENVALUE ASYMPTOTICS FOR ZAKHAROV–SHABAT SYSTEMS WITH LONG–RANGE POTENTIALS

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Abstract. We study the spectrum of Zakharov-Shabat (ZS) systems with long-range potentials that have infinitely many purely imaginary eigenvalues accumulating at the origin. We consider $N(s)$, the number of imaginary eigenvalues with imaginary part strictly larger than s. If the potential $q(t)$ is positive and falls off like $|t|^{-\gamma}$, $0<\gamma\leqslant 1$, and satisfies some additional technical conditions, we prove that $N(s)\sim\pi^{-1}\int_{\{t:q(t)>s\}}(q(t)^2-s^2)^{1/2}\,dt$. Therefore, we have a connection with the well known phase volume integral from quantum mechanics for the number of eigenvalues less than $-s^2$ for a Schrödinger operator with potential $-q(t)^2$. However, in contrast to Schrödinger operators, a major difficulty arises from the fact that the ZS system, since it is nonselfadjoint, may have eigenvalues that are not algebraically simple. We will pay special attention to this difficulty and prove a new result (Theorem **??**) which says that nonsimple eigenvalues do not occur if s is sufficiently small.

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