

SHARPENING SOME CLASSICAL NUMERICAL RADIUS INEQUALITIES

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Abstract. New upper and lower bounds for the numerical radii of Hilbert space operators are given. Among our results, we prove that if $A \in \mathcal{B}(\mathcal{H})$ is a hyponormal operator, then for all non-negative non-decreasing operator convex f on $[0, \infty)$, we have

$$f(\omega(A)) \leq \frac{1}{2} \left\| \left\| f\left(\frac{1}{1 + \frac{\xi_{|A|}^2}{8}} |A|\right) + f\left(\frac{1}{1 + \frac{\xi_{|A^*|}^2}{8}} |A^*|\right) \right\| \right\|,$$

where $\xi_{|A|} = \inf_{\|x\|=1} \left\{ \frac{((|A| - |A^*|)x, x)}{((|A| + |A^*|)x, x)} \right\}$. Our results refine and generalize earlier inequalities for hyponormal operator.

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