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SPECTRAL ANALYSIS OF THE DIRAC OPERATOR ON A 3–SPHERE

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Abstract. We study the (massless) Dirac operator on a 3-sphere equipped with Riemannian metric. For the standard metric the spectrum is known. In particular, the eigenvalues closest to zero are the two double eigenvalues $+3/2$ and $-3/2$. Our aim is to analyse the behaviour of eigenvalues when the metric is perturbed in an arbitrary smooth fashion from the standard one. We derive explicit perturbation formulae for the two eigenvalues closest to zero, taking account of the second variations. Note that these eigenvalues remain double eigenvalues under perturbations of the metric: they cannot split because of a particular symmetry of the Dirac operator in dimension three (it commutes with the antilinear operator of charge conjugation). Our perturbation formulae show that in the first approximation our two eigenvalues maintain symmetry about zero and are completely determined by the increment of Riemannian volume. Spectral asymmetry is observed only in the second approximation of the perturbation process. As an example we consider a special family of metrics, the so-called generalized Berger spheres, for which the eigenvalues can be evaluated explicitly.

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