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ON SOME p–ALMOST HADAMARD MATRICES

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Abstract. Let $\mathrm{M}(n,\mathbb{R})$ be the space of all real valued $n \times n$ matrices and $\mathrm{O}(n,\mathbb{R})$ be the orthogonal group. A square matrix $\mathrm{H}_n \in \mathrm{M}(n,\mathbb{R})$ is called "almost Hadamard" if $\mathrm{U}_n := \mathrm{H}_n/\sqrt{n}$ is orthogonal, and locally maximizes the 1-norm on $\mathrm{O}(n,\mathbb{R})$. The matrix H_n is "p-almost Hadamard" if it maximizes the p-norm on $\mathrm{O}(n,\mathbb{R})$ for $p \in [1,2)$ and minimizes the p-norm on $\mathrm{O}(n,\mathbb{R})$ for $p \in (2,\infty]$. In this work, we consider the Conjecture 4.4 stated in [8] and discuss its truth content. For $n \in \mathbb{N}\backslash\{2\}$, we show that the matrix

$$\mathrm{K}_n := \frac{1}{\sqrt{n}}\begin{pmatrix} 2-n & 2 & \cdots & 2 \\ 2 & 2-n & \cdots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & \cdots & 2-n \end{pmatrix}$$

is p-almost Hadamard, for any $p \in (2,\infty)$ such that

$$(p-1)\left[(n-2)2^p + (n-2)^2 2^{p-2} + 2^2(n-2)^{p-2}\right] > (n-2)^p + 2^p(n-1).$$

We also establish that for any $p \in [1,2)$ and $n \in \mathbb{N}\backslash\{2\}$, K_n is p-almost Hadamard and hence the Conjecture is valid for this case. Finally, we give some particular examples of p-almost Hadamard matrices of different orders, incorporating conference and weighing matrices.

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