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SELF–ADJOINT OPERATORS AND THE GENERAL GKN–EM THEOREM

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Abstract. We construct self-adjoint operators in the direct sum of a complex Hilbert space H and a finite dimensional complex inner product space W. The operator theory developed in this paper for the Hilbert space $H \oplus W$ is originally motivated by some fourth-order differential operators, studied by Everitt and others, having orthogonal polynomial eigenfunctions. Generated by a closed symmetric operator T_0 in H with equal and finite deficiency indices and its adjoint T_1, we define *families* of minimal operators $\{\widehat{T}_0\}$ and maximal operators $\{\widehat{T}_1\}$ in the extended space $H \oplus W$ and establish, using a recent theory of complex symplectic geometry, developed by Everitt and Markus, a characterization of self-adjoint extensions of $\{\widehat{T}_0\}$ when the dimension of the extension space W is not greater than the deficiency index of T_0. A generalization of the classical Glazman-Krein-Naimark (GKN) theorem - called the GKN-EM theorem to acknowledge the work of Everitt and Markus - is key to finding these self-adjoint extensions in $H \oplus W$. We consider several examples to illustrate our results.

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