SELFADJOINT OPERATORS, NORMAL OPERATORS, AND CHARACTERIZATIONS

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Abstract. Let $\mathcal{B}(H)$ be the $C^*$-algebra of all bounded linear operators acting on a complex separable Hilbert space $H$. We shall show that:

1. The class of all selfadjoint operators in $\mathcal{B}(H)$ multiplied by scalars is characterized by
   \[ \forall X \in \mathcal{B}(H), \| S^2X + XS^2 \| \geq 2 \| XS \|, \quad (S \in \mathcal{B}(H)). \]

2. The class of all normal operators in $\mathcal{B}(H)$ is characterized by each of the three following properties (where $D_S = S^*S - SS^*$, for $S \in \mathcal{B}(H)$),
   \begin{enumerate}
   \item[(i)] $\forall X \in \mathcal{B}(H), \| S^2X \| + \| XS^2 \| \geq 2 \| XS \|, (S \in \mathcal{B}(H))$,
   \item[(ii)] $S^*D_SS = 0 = SD_SS^*$, $(S \in \mathcal{B}(H))$,
   \item[(iii)] $S^*D_SS \geq 0 \geq SD_SS^*$, $(S \in \mathcal{B}(H))$.
   \end{enumerate}


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REFERENCES