CONSTANT NORMS AND NUMERICAL RADII OF MATRIX POWERS

HWA-LONG GAU, KUO-ZHONG WANG AND PEI YUAN WU

Abstract. For an $n$-by-$n$ complex matrix $A$, we consider conditions on $A$ for which the operator norms $\|A^k\|$ (resp., numerical radii $w(A^k)$), $k \geq 1$, of powers of $A$ are constant. Among other results, we show that the existence of a unit vector $x$ in $\mathbb{C}^n$ satisfying $|\langle A^kx, x \rangle| = w(A^k) = w(A)$ for $1 \leq k \leq 4$ is equivalent to the unitary similarity of $A$ to a direct sum $\lambda B \oplus C$, where $|\lambda| = 1$, $B$ is idempotent, and $C$ satisfies $w(C^k) \leq w(B)$ for $1 \leq k \leq 4$. This is no longer the case for the norm: there is a 3-by-3 matrix $A$ with $\|A^kx\| = \|A^k\| = \sqrt{2}$ for some unit vector $x$ and for all $k \geq 1$, but without any nontrivial direct summand. Nor is it true for constant numerical radii without a common attaining vector. If $A$ is invertible, then the constancy of $\|A^k\|$ (resp., $w(A^k)$) for $k = \pm 1, \pm 2, \ldots$ is equivalent to $A$ being unitary. This is not true for invertible operators on an infinite-dimensional Hilbert space.


Keywords and phrases: Operator norm, numerical radius, idempotent matrix, irreducible matrix.

REFERENCES