Operators and Matrices
Volume 13, Number 4 (2019), 1091–1124
doi:10.7153/oam-2019-13-74

EXAMPLES FOR THE QUANTUM KIPPENHAHN THEOREM

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Abstract. Semidefinite programming optimises a linear objective function over a spectrahedron, and is one of the major advances of mathematical optimisation. Spectrahedra are described by linear pencils, which are linear matrix polynomials with hermitian matrix coefficients. Our focus will be on dimension-free linear pencils where the variables are permitted to be hermitian matrices. A major question on linear pencils, and matrix theory in general, is Kippenhahn's conjecture. The conjecture states that given a linear pencil $xH+yK$ if the hermitian matrices H and K generate the full matrix algebra, then the pencil must have at least one simple eigenvalue for some x and y. The conjecture is known to be false, via a single counterexample due to Laffey. A dimension-free version of the conjecture, known as the Quantum Kippenhahn theorem, has recently been proven true non-constructively. We present a novel family of counterexamples to Kippenhahn's conjecture, and use this family to construct concrete examples of the Quantum Kippenhahn theorem.

Mathematics subject classification (2010): 15A22, 15A42, 46L52, 90C22.
Keywords and phrases: Free analysis, non-commutative algebra, linear pencil, double eigenvalues, Kippenhahn conjecture.

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