LAPLACIANS ON BIPARTITE METRIC GRAPHS

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Abstract. We study spectral properties of the standard (also called Kirchhoff) Laplacian and the anti-standard (or anti-Kirchhoff) Laplacian on a finite, compact metric graph. We show that the positive eigenvalues of these two operators coincide whenever the graph is bipartite; this leads to a precise relation between their eigenvalues enumerated with multiplicities and including the possible eigenvalue zero. Several spectral inequalities for, e.g., trees are among the consequences of this. In the second part we study inequalities between standard and Dirichlet eigenvalues in more detail and expose another connection to bipartiteness.

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