

\mathcal{C} -SYMMETRIC SECOND ORDER DIFFERENTIAL OPERATORS

HORST BEHNCKE AND DON HINTON

Abstract. We consider a \mathcal{C} -Symmetric second order linear differential operator on a half interval or the real line. We determine the spectrum and construct the resolvent and m -function. In addition we analyze the resolvent and m -function near their poles. Under the conditions of Theorem 2.2 we prove the essential spectrum is empty, and the operator has a compact resolvent. Integral conditions on the operator coefficients are given in Theorem 3.4 for the operator to be Hilbert-Schmidt. These conditions are new even in the selfadjoint case. This analysis is based on asymptotic integration. A central role is played by the Titchmarsh-Weyl m -function which is defined by square integrable functions and not by a nesting circle analysis.

Mathematics subject classification (2010): Primary 34L05, 34B20, 34B27, 34B40, 34B60.

Keywords and phrases: m -functions, singular operators, essential spectrum, non-selfadjoint operators, \mathcal{C} -Symmetric operators, Green's functions.

REFERENCES

- [1] M. ABRAMOWITZ AND I. STEGUN, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover Publications, New York, 1965.
- [2] C. AHLBRANDT, D. HINTON, AND R. LEWIS, *The effect of variable change on oscillation and disconjugacy criteria with applications to spectral theory and asymptotic theory*, J. Math. Anal. and Appl. **81** (1981), 243–277.
- [3] H. BEHNCKE AND D. HINTON, *Transformation theory of symmetric differential expressions*, Advances in Diff. Eqs. **11** (2006), 601–626.
- [4] H. BEHNCKE, *Asymptotically constant linear systems*, Proc. AMS **138** (2010), 1387–1393.
- [5] H. BEHNCKE, D. HINTON, AND C. REMLING, *The spectrum of differential equations of order $2n$ with almost constant coefficients*, J. of Diff. Equations **175** (2001), 130–162.
- [6] H. BEHNCKE AND D. B. HINTON, *Hamiltonian Systems with Almost Constant Coefficients*, J. of Diff. Equations **250** (2011), 1403–1426.
- [7] H. BEHNCKE AND D. B. HINTON, *Two Singular Point Linear Hamiltonian Systems with an Interface Condition*, Math. Nachr **283** (2010), 365–378.
- [8] H. BEHNCKE AND D. B. HINTON, *A Class of Differential Operators with Complex Coefficients and Compact Resolvent*, Differential and Integral Equations **31** (2018), 375–402.
- [9] H. BEHNCKE AND D. B. HINTON, *\mathcal{C} -Symmetric Hamiltonian Systems with Almost Constant Coefficients*, Journal of Spectral Theory **9** (2019), 513–546.
- [10] B. M. BROWN, D. MCCORMACK, W. D. EVANS, AND M. PLUM, *On the Essential Spectrum of Second-order Differential Equations with Complex Coefficients*, Proc. Royal Soc. London A **455** (1999), 1235–1257.
- [11] B. M. BROWN, W. D. EVANS, AND M. PLUM, *Titchmarsh-Sims-Weyl Theory for Complex Hamiltonian Systems*, Proc. London Math. Soc. **87** (2003), 419–450.
- [12] Z. CHEN AND C. LIN, *On Algebraic Multiplicity of (Anti)Periodic Eigenvalues of Hill's Equations*, Proc. AMS **146** (2018), 3039–3047.
- [13] E. CODDINGTON AND N. LEVINSON, *Theory of Ordinary Differential Equations*, McGraw Hill, York, PA, 1955.
- [14] N. DUNFORD AND J. T. SCHWARZ, *Linear Operators III*, Interscience, New York, 1963.
- [15] M. S. P. EASTHAM, *The Asymptotic Solution of Linear Differential Systems*, London Mathematical Monographs, **4**, 1989.

- [16] M. S. P. EASTHAM AND H. KALF, *Schrodinger-type Equations with Continuous Spectra*, Pitman, London, 1982.
- [17] S. GARCIA, E. PRODAN, AND M. PUTINAR, *Mathematical and physical aspects of complex symmetric operators*, J. Phys. A: Math. Theor. **47** (2014), 54 pp.
- [18] I. M. GLAZMAN, *Direct Methods of Qualitative Spectral Analysis of Singular Differential Operators*, Israel Program for Sc. Tranl. Jerusalem, 1963.
- [19] I. GIORDANELLI AND G. GRAF, *The real spectrum of the imaginary cubic oscillator*, Ann. Henri Poincaré **16** (2015), 99–112.
- [20] S. GOLDBERG, *Unbounded Linear Operators*, McGraw Hill, New York 1966.
- [21] W. A. HARRIS AND D. A. LUTZ, *Asymptotic Integration of Adabatic Oscillators*, J. Math. Anal. Appl. **51** (1975), 76–93.
- [22] I. HERBST, *Dilation analyticity in constant electric field, I. The two body problem*, Comm. Math. Physics **64** (1979), 279–298.
- [23] N. JU AND S. WIGGINS, *On Roughness of Exponential Dichotomy*, J. Math. Anal. Appl. **262** (2001), 39–49.
- [24] T. KATO, *Perturbation theory for linear operators*, Springer-Verlag, New York, 1966.
- [25] R. KAUFFMAN, T. READ, AND A. ZETTL, *The deficiency index problem for powers of ordinary differential equations*, Lecture Notes in Mathematics **621**, Springer-Verlag, New York, 1977.
- [26] R. R. D. KEMP, *A singular boundary value problem for a non-self-adjoint differential operator*, Canadian J. Math. **13** (1958), 447–462.
- [27] I. KNOWLES, *On the Boundary Conditions Characterizing J-Selfadjoint Extensions of J-Symmetric Operators*, J. of Differential Equations **40** (1981), 193–216.
- [28] J. LOCKER, *Spectral Theory on Non-Selfadjoint Two-Point Differential Operators*, Mathematical Surveys and Monographs, vol. 73, AMS, Providence, 2000.
- [29] V. LYANTSE, *The eigenfunction expansion of a non-selfadjoint differential operator with spectral singularities* (Russian), Dokl. Akad. Nauk SSSR **149** (1963), 256–259.
- [30] J. B. MCLEOD, *Square-Integrable Solutions of a Second-Order Differential Equation with Complex Coefficients*, Quart. J. Math. Oxford (2) **13**, (1962), 129–133.
- [31] M. MUZZOLINI, *Titchmarsh-Sims-Weyl Theory for Complex Hamiltonian Systems of Arbitrary Order*, J. London Math Soc. **84** (2011), 159–182.
- [32] G. O. OKIKIOLU, *Aspects of the Theory of bounded Integral Operators in L_p -Spaces*, Academic Press, London, New York, 1971.
- [33] B. PAVLOV, *On the non-selfadjoint operator $-y'' + q(x)y$ on a semiaxis* (Russian), Dokl. Akad. Nauk SSSR **141** (1961), 807–810.
- [34] C. REMLING, *Spectral Analysis of Higher Order Differential Operators I: General Properties of the M-Function*, J. London Math. Soc. **58** (1998), 361–380.
- [35] B. RUSSO, *On the Hausdorff-Young Theorem for Integral Operators*, Pacific J. Mathematics **68** (1977), 241.
- [36] A. SIMS, *Secondary conditions for linear differential equations of the second order*, J. Math. Mech. **6** (1957), 247–285.
- [37] J. VON NEUMANN AND E. P. WIGNER, *Über merk würdige diskrete Eigenwetze*, Z. Physics **30** (1929), 465–467.