# EXPLICIT KREIN RESOLVENT IDENTITIES FOR SINGULAR STURM—LIOUVILLE OPERATORS WITH APPLICATIONS TO BESSEL OPERATORS

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*Abstract.* We derive explicit Krein resolvent identities for generally singular Sturm–Liouville operators in terms of boundary condition bases and the Lagrange bracket. As an application of the resolvent identities obtained, we compute the trace of the resolvent difference of a pair of self-adjoint realizations of the Bessel expression $-d^2/dx^2 + (v^2 - (1/4))x^{-2}$ on $(0,\infty)$ for values of the parameter $v \in [0,1)$ and use the resulting trace formula to explicitly determine the spectral shift function for the pair.

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