# GRAPH COMPLEMENT CONJECTURE
# FOR CLASSES OF SHADOW GRAPHS

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*Abstract.* The real minimum semidefinite rank of a graph $G$, denoted $\mathrm{mr}_+^{\mathbb{R}}(G)$, is defined to be the minimum rank among all real symmetric positive semidefinite matrices whose zero/nonzero pattern corresponds to the graph $G$. The inequality $\mathrm{mr}_+^{\mathbb{R}}(G) + \mathrm{mr}_+^{\mathbb{R}}(\overline{G}) \leqslant |G| + 2$ is called the graph complement conjecture, denoted $GCC_+$, where $\overline{G}$ is the complement of $G$ and $|G|$ is the number of vertices in $G$. A known definition of shadow graph $S(G)$ and a variant of this definition denoted $\mathrm{Shad}(G)$ are given. It is shown that $S(G)$ satisfies $GCC_+$ when $G$ is a tree or a unicyclic graph or a complete graph. Under additional conditions on $\overline{G}$, it is shown that $S(G)$ satisfies $GCC_+$ when $G$ is a $k$-tree or a chordal graph. Moreover, whenever $G$ satisfies $GCC_+$ and $\overline{G}$ does not contain any isolated vertices, it is shown that $\mathrm{Shad}(G)$ satisfies $GCC_+$.

*Mathematics subject classification* (2020): 05C50, 15A03, 15A18, 15B57.
*Keywords and phrases*: Shadow graphs, minimum semidefinite rank, graph complement conjecture.

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