

## FROBENIUS NORM INEQUALITIES OF COMMUTATORS BASED ON DIFFERENT PRODUCTS

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*Abstract.* The difference  $AB - BA$  of two matrices  $A$  and  $B$  is called the commutator (or Lie product). In this paper, we are concerned with inequalities for the Frobenius norm of commutators based on other products, including the Kronecker product, the Khatri-Rao product, the contracted product, and the T-product. We also study the characterization of their corresponding maximal pairs.

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