

## INTERPLAY OF SIMPLE AND SELFADJOINT-IDEAL SEMIGROUPS IN $B(H)$

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*Abstract.* This paper investigates a question of Radjavi: Which multiplicative semigroups in  $B(H)$  have all their ideals selfadjoint (called herein selfadjoint-ideal (SI) semigroups)? We proved this property is a unitary invariant for  $B(H)$ -semigroups, which invariant we believe is new.

We characterize those SI semigroups  $\mathcal{S}$  singly generated by  $T$ , for  $T$  a normal operator and for  $T$  a rank one operator. When  $T$  is nonselfadjoint and normal or rank one:  $\mathcal{S}$  is an SI semigroup if and only if it is simple, except in one special rank one partial isometry case when our characterization yields  $\mathcal{S}$  that are SI but not simple. So SI and simplicity are not equivalent notions. When  $T$  is selfadjoint, it is straightforward to see that  $\mathcal{S}$  is always an SI semigroup, but we prove by examples they may or may not be simple, but for this case we do not have a characterization.

The study of SI semigroups involves solving certain operator equations in the semigroups. A central theme of this paper is to study when and when not SI is equivalent to simple.

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