OPERATOR SPLITTING FOR ABSTRACT CAUCHY PROBLEMS WITH DYNAMICAL BOUNDARY CONDITIONS

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Abstract. In this work we study operator splitting methods for a certain class of coupled abstract Cauchy problems, where the coupling is such that one of the sub-problems prescribes a "boundary type" extra condition for the other one. The theory of one-sided coupled operator matrices provides an excellent framework to study the well-posedness of such problems. We show that with this machinery even operator splitting methods can be treated conveniently and rather efficiently. We consider three specific examples: the Lie (sequential), the Strang, and the weighted splitting, and prove the convergence of these methods along with error bounds under fairly general assumptions. Simple numerical examples show that the obtained theoretical bounds can be computationally realised.

Mathematics subject classification (2020): 47D06, 47N40, 34G10, 65J08, 65M12, 65M15.

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