THE APS–INDEX AND THE SPECTRAL FLOW

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Abstract. We study the Atiyah-Patodi-Singer (APS) index, and its equality to the spectral flow, in an abstract, functional analytic setting. More precisely, we consider a (suitably continuous or differentiable) family of self-adjoint Fredholm operators $A(t)$ on a Hilbert space, parametrised by t in a finite interval. We then consider two different operators, namely $D := \frac{d}{dt} + A$ (the abstract analogue of a Riemannian Dirac operator) and $D := \frac{d}{dt} - iA$ (the abstract analogue of a Lorentzian Dirac operator). The latter case is inspired by a recent index theorem by Bär and Strohmaier (Amer. J. Math. 141 (2019), 1421–1455) for a Lorentzian Dirac operator equipped with APS boundary conditions. In both cases, we prove that the Fredholm index of the operator D equipped with APS boundary conditions is equal to the spectral flow of the family $A(t)$.

Mathematics subject classification (2020): 47A53, 58J20, 58J30.
Keywords and phrases: Atiyah-Patodi-Singer index, Fredholm pairs of projections, spectral flow.

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