Operators and Matrices
Volume 16, Number 3 (2022), 775–792
doi:10.7153/oam-2022-16-55

SINGULAR VALUE STRUCTURE OF REAL MATRICES WHICH CAN BE EXPRESSED AS A LINEAR COMBINATION OF TWO ORTHOGONAL MATRICES

ZHENG LI*, TIE ZHANG AND CHANG-JUN LI

Abstract. Can every real matrix be expressed as a linear combination of a certain number of orthogonal matrices? What is the smallest number of these orthogonal matrices? These elegant and interesting questions were initially raised by Zhan [7]. Soon afterwards, Li and Poon [4] proved that k_{min}, the smallest number of these orthogonal matrices, is not greater than 4. These classic results inspire us to further explore the improvement or supplement of this theory.

We investigate some fundamental properties of $\mathscr{A}_n(k)$, which is the set of all $n \times n$ real matrices that can be expressed as a linear combination of k orthogonal matrices. Furthermore, we characterize the singular value structure of matrices in set $\mathscr{A}_n(2)$ and the block structure of related orthogonal matrices. We obtain an equivalent condition and some sufficient or necessary conditions of $A \in \mathscr{A}_n(2)$. Based on these results, we demonstrate the existence of matrices that are not in set $\mathscr{A}_n(2)$, and prove that $k_{min} > 2$ (for $n \geqslant 3$).

Mathematics subject classification (2020): 15A16, 15A18.
Keywords and phrases: Orthogonal matrices, linear combination, singular value, additive decomposition, cone.

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Paper OaM-16-55