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GENERALIZED CHOI–KRAUS DILATIONS OF LINEAR MAPS BETWEEN MATRIX ALGEBRAS

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Abstract. By the generalized Stinespring's dilation theorem, every linear map between two matrix algebras M_n and M_d has $*$-homomorphism dilation due to the fact that such a map is always completely bounded. In fact, since every such a map has a generalized Choi-Kraus representation $\varphi(X) = \sum_{k=1}^{L} A_k X B_k^*$, it automatically induces a $*$-homomorphism dilation by the representation matrix system $\{A_k, B_k\}$, which we call it the generalized Choi-Kraus dilation for a linear map, and the Choi-Kraus dilation when $A_k = B_k$ for a completely positive (CP) map. The purpose of this paper is to examine the connections between the generalized Choi-Kraus dilations with other well-established dilations including the universal dilation. We prove that any linearly minimal $*$-homomorphism dilation is equivalent to a linearly minimal generalized Choi-Kraus dilation, and present a necessary and sufficient condition for the equivalence of two linearly minimal generalized Choi-Kraus dilations. While all the linearly minimal Choi-Kraus dilations for a CP map are unitarily equivalent, the linearly minimal generalized Choi-Kraus dilations, even for a CP map, are not necessarily equivalent. In fact, a linear map admits only one equivalent class of linearly minimal generalized Choi-Kraus dilations if and only if its generalized Choi matrix has full rank.

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