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ON 2×2 POSITIVE MATRICES OF τ–MEASURABLE OPERATORS

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Abstract. Let $\mathscr{M}$ be a semi-finite von Neumann algebra. We proved the following inequalities are hold and equivalent:

(i) If $x,y\in L_{\log_+}(\mathscr{M})$ are self-adjoint operators such that $\pm y\leqslant x$, then $y\preccurlyeq_{\log} x$.

(ii) If $a,b\in\mathscr{M}$, $x,y\in L_{\log_+}(\mathscr{M})$ and $\begin{pmatrix} x & z\\ z^* & y\end{pmatrix}\geqslant 0$, then

$$a^*zb+b^*z^*a\preccurlyeq_{\log} a^*xa+b^*yb.$$

(iii) If $x,y,z\in L_{\log_+}(\mathscr{M})$ and $\begin{pmatrix} x & z\\ z^* & y\end{pmatrix}\geqslant 0$, then $z^*+z\preccurlyeq_{\log} x+y$.

(iv) If $x,y\in L_{\log_+}(\mathscr{M})$ are positive operators, then $x-y\preccurlyeq_{\log} x+y$.

(v) If $x,y,z\in L_{\log_+}(\mathscr{M})$ and $\begin{pmatrix} x & z\\ z^* & y\end{pmatrix}\geqslant 0$, then $z^*\oplus z\preccurlyeq_{\log} x\oplus y$.

(vi) If $x,y\in L_{\log_+}(\mathscr{M})$ are normal operators and $z\in L_{\log_+}(\mathscr{M})$ is positive operator, then for any contraction $a\in\mathscr{M}$,

$$|za(x+y)a^*z|\preccurlyeq_{\log} za(|x|+|y|)a^*z.$$

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