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FURTHER REFINEMENTS OF DAVIS–WIELANDT RADIUS INEQUALITIES

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Abstract. Suppose T,S are bounded linear operators on a complex Hilbert space. We show that the Davis-Wielandt radius $dw(\cdot)$ satisfies the following inequalities

$$\begin{aligned} dw(T+S) &\leqslant \sqrt{2\big(dw^2(T)+dw^2(S)\big)+6\big\||T|^4+|S|^4\big\|} \\ &\leqslant 2\sqrt{2}\sqrt{dw^2(T)+dw^2(S)} \\ &\leqslant 2\sqrt{2}\big(dw(T)+dw(S)\big). \end{aligned}$$

From the third inequality we obtain the following lower and upper bounds for the Davis-Wielandt radius $dw(T)$ of the operator T:

$$\begin{aligned} dw(T) &\geqslant \frac{1}{4\sqrt{2}}\max\big\{dw\big(2Re(T)\big),dw\big(2Im(T)\big)\big\}, \\ dw(T) &\leqslant 2\sqrt{2}\left(dw\big(Re(T)\big)+dw\big(Im(T)\big)\right). \end{aligned}$$

Further, we develop several new lower and upper bounds for the Davis-Wielandt radius of the operator T which improve the existing ones. Application of these bounds are also provided.

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