Operators and Matrices
Volume 17, Number 3 (2023), 779–791
doi:10.7153/oam-2023-17-51

THE INFINITE DIMENSIONAL PERFECT–MIRSKY CONJECTURE

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Abstract. The spectrum of an infinite-dimensional doubly stochastic matrix, when considered as a bounded operator on the sequence space ℓ^p with $1 \leqslant p < \infty$, is contained within the closed unit disc $\mathbb{D}$. In our work, we present an infinite doubly stochastic matrix that exhibits the entire closed unit disc as its spectrum. However, we prove that the points $e^{i\pi r}$, where r is an irrational real number, cannot serve as eigenvalues for any doubly stochastic matrices, be it finite or infinite in size. On the other hand, we show that every other point within the closed unit disc can indeed be an eigenvalue of an infinite-dimensional doubly stochastic matrix. In fact, we construct a specific example of an infinite doubly stochastic matrix whose point spectrum precisely consists of $\mathbb{D} \cup \{e^{i\pi r} : r \in \mathbb{Q}\}$. Additionally, we investigate the behavior of doubly stochastic matrices in the context of the sequence space ℓ^∞, highlighting the contrasts with the ℓ^p setting for $1 \leqslant p < \infty$.

Mathematics subject classification (2020): 15B48, 15B51, 15A45, 47A50.
Keywords and phrases: Doubly stochastic matrices, doubly sub-stochastic matrices, spectrum, point spectrum.

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