INNER PRODUCT INEQUALITIES THROUGH
CARTESIAN DECOMPOSITION WITH APPLICATIONS
TO NUMERICAL RADIUS INEQUALITIES

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Abstract. This paper intends to show several inner product inequalities using the Cartesian decomposition of the operator. We utilize the obtained results to get norm and numerical radius inequalities. Our results extend and improve some earlier inequalities. Among other inequalities, it is revealed that if T is a $n \times n$ complex matrix with the imaginary part $\Im T = \frac{T - T^*}{2i}$, then

$$\frac{1}{2} \max \left(\left\| TT^* - i\Im T^2 \right\|^{\frac{1}{2}}, \left\| T^*T + i\Im T^2 \right\|^{\frac{1}{2}} \right) \leqslant \omega(T)$$

which is a significant improvement of the classical inequality $\frac{1}{2} \|T\| \leqslant \omega(T)$.

REFERENCES

[1] R. BHATIA, F. KITTANEH, *On the singular values of a product of operators*, SIAM J. Matrix Anal. Appl., **11** (1990), 272–277.

[2] P. BHUNIA, K. PAUL, *New upper bounds for the numerical radius of Hilbert space operators*, Bull. Sci. Math., **167** (2021), Paper No. 102959, 11 pp.

[3] P. BHUNIA, K. PAUL, *Refinement of numerical radius inequalities of complex Hilbert space operators*, Acta Sci. Math. (Szeged), (2023). https://doi.org/10.1007/s44146-023-00070-1

[4] M. L. BUZANO, *Generalizzazione della diseguaglianza di Cauchy–Schwarz'*, Rend. Sem. Mat. Univ. Politech. Torino., **31** (1971/73), 405–409; (1974) (in Italian).

[5] M. AL-DOLAT, I. JARADAT, AND B. AL-HUSBAN, *A novel numerical radius upper bounds for* 2×2 *operator matrices*, Linear Multilinear Algebra., **70** (6) (2022), 1173–1184.

[6] S. S. DRAGOMIR, *Power inequalities for the numerical radius of a product of two operators in Hilbert spaces*, Sarajevo J. Math., **5** (18) (2009), 269–278.

[7] S. S. DRAGOMIR, *Some inequalities for the Euclidean operator radius of two operators in Hilbert spaces*, Linear Algebra Appl., **419** (2006), 256–264.

[8] M. EL-HADDAD, F. KITTANEH, *Numerical radius inequalities for Hilbert space operators. II*, Studia Math., **182** (2) (2007), 133–140.

[9] T. FURUTA, *Invitation to Linear Operators*, Taylor and Francis, London, 2001.

[10] I. H. GÜMÜŞ, H. R. MORADI, M. SABABHEH, *On positive and positive partial transpose matrices*, Electron. J. Linear Algebra., **38** (2022), 792–802.

[11] P. R. HALMOS, *A Hilbert Space Problem Book*, 2nd ed., Springer, New York, 1982.

[12] M. HASSANI, M. E. OMIDVAR, AND H. R. MORADI, *New estimates on numerical radius and operator norm of Hilbert space operators*, Tokyo J. Math., **44** (2) (2021), 439–449.

[13] O. HERZALLAH, F. KITTANEH, AND K. SHEBRAWI, *Numerical radius inequalities for certain* 2×2 *operator matrices*, Integr. Equ. Oper. Theory., **71** (2011), 129–147.

[14] Z. Heydarbeygi, M. Sababheh, and H. R. Moradi, *A convex treatment of numerical radius inequalities*, Czech Math J., **72** (2022), 601–614.

[15] F. Kittaneh, *Numerical radius inequalities associated with the Cartesian decomposition*, Math. Inequal. Appl., **18** (3) (2015), 915–922.

[16] F. Kittaneh, *Numerical radius inequalities for Hilbert space operators*, Studia Math., **168** (1) (2005), 73–80.

[17] F. Kittaneh, H. R. Moradi, and M. Sababheh, *Sharper bounds for the numerical radius*, Linear Multilinear Algebra. https://doi.org/10.1080/03081087.2023.2177248.

[18] M. Lin, D. Zhou, *Norm inequalities for accretive-dissipative operator matrices*, J. Math. Anal. Appl., **407** (2013), 436–442.

[19] J. S. Matharu, J. S. Aujla, *Some inequalities for unitarily invariant norms*, Linear Algebra Appl., **436** (2012), 1623–1631.

[20] F. P. Najafabadi, H. R. Moradi, *Advanced refinements of numerical radius inequalities*, International Journal of Mathematical Modelling & Computations., **11** (4) (2021), 1–10.

[21] S. Tafazoli, H. R. Moradi, S. Furuichi, and P. Harikrishnan, *Further inequalities for the numerical radius of Hilbert space operators*, J. Math. Inequal., **13** (4) (2019), 955–967.

[22] T. Yamazaki, *On upper and lower bounds of the numerical radius and an equality condition*, Studia Math. 178 (2007), 83–89.

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