Operators
and
Matrices
Volume 18, Number 1 (2024), 163–172
doi:10.7153/oam-2024-18-11

JOINT SPECTRUM SHRINKING MAPS ON PROJECTIONS

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Abstract. Let $\mathscr{H}$ be a finite dimensional complex Hilbert space with dimension $n \geqslant 3$ and $\mathscr{P}(\mathscr{H})$ the set of projections on $\mathscr{H}$. Let $\varphi : \mathscr{P}(\mathscr{H}) \to \mathscr{P}(\mathscr{H})$ be a surjective map. We show that φ shrinks the joint spectrum of any two projections if and only if it is induced by a semilinear automorphism on $\mathscr{H}$. In addition, φ shrinks the joint spectrum of I, P, Q for any two projections $P, Q \in \mathscr{P}(\mathscr{H})$ if and only if it is induced by a unitary or an anti-unitary. Assume that ϕ is a surjective map on the Grassmann space of rank one projections. We show that ϕ is joint spectrum shrinking for any n rank one projections if and only if it is induced by a semilinear automorphism on $\mathscr{H}$. Moreover, for any $k > n$, ϕ is joint spectrum shrinking for any k rank one projections if and only if it is induced by a unitary or an anti-unitary.

Mathematics subject classification (2020): Primary 47B49; Secondary 47A25.
Keywords and phrases: Joint spectrum preserving, joint spectrum shrinking, Kaplansky Problem, projections.

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