Operators
and
Matrices
Volume 18, Number 2 (2024), 273–293
doi:10.7153/oam-2024-18-17

ON MINIMAL SMALLEST SINGULAR VALUE OF SUBFRAMES FOR SIGNAL RECOVERY

YANG LIU

Abstract. In this paper, we mainly study the smallest singular value of submatrices consisting of row vectors bounded by 1, and we establish that the minimal smallest singular value of submatrices of matrices of size $n+1$ times n consisting of row vectors bounded by 1 is equal to $\frac{1}{\sqrt{n}}$ if and only if the rows of $\mathrm{diag}(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_{n+1})A$ are the coordinates of the $n+1$ vertices of a regular n-simplex on the unit $(n-1)$-sphere S^{n-1} in $\mathbb{R}^n$ for some $(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_{n+1})\in\{-1,1\}^{n+1}$. Moreover, we establish that the minimal smallest singular value of submatrices of matrices of size n times 2 consisting of row vectors bounded by 1 is sharply bounded above by $\sqrt{2}\sin\frac{\pi}{2n}$, and furthermore, this bound is achieved if and only if the rows of $\mathrm{diag}(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)A$ are the coordinates of n adjacent vertices of a regular $2n$-gon on the unit circle S^1 in $\mathbb{R}^2$ for some $(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)\in\{-1,1\}^n$. Additionally, we show that the equiangular frames in the projective spaces do not form the matrices in the general dimensions with the optimal smallest singular value of the submatrices, contrary to the case of matrices of dimension $n+1$ by n or negative to the conjectures based on the phenomena in the low dimensions.

Mathematics subject classification (2020): 05C50, 15A60, 33F05.
Keywords and phrases: Matrix analysis, singular values, signal processing, optimization.

REFERENCES

[1] MICHAL AHARON, MICHAEL ELAD, AND ALFRED BRUCKSTEIN, *K-svd: An algorithm for designing overcomplete dictionaries for sparse representation*, IEEE Transactions on signal processing, **54** (11): 4311–4322, 2006.

[2] AKRAM ALDROUBI, I. KRISHTAL, AND SUI TANG, *Phaseless reconstruction from space-time samples*, Applied and Computational Harmonic Analysis, **48** (1): 395–414, 2020.

[3] LORNE APPLEBAUM, STEPHEN D. HOWARD, STEPHEN SEARLE, AND ROBERT CALDERBANK, *Chirp sensing codes: Deterministic compressed sensing measurements for fast recovery*, Applied and Computational Harmonic Analysis, **26** (2): 283–290, 2009.

[4] Z. D. BAI AND Y. Q. YIN, *Limit of the smallest eigenvalue of a large dimensional sample covariance matrix*, The annals of Probability, pages 1275–1294, 1993.

[5] RADU BALAN AND YANG WANG, *Invertibility and robustness of phaseless reconstruction*, Applied and Computational Harmonic Analysis, **38** (3): 469–488, 2015.

[6] GILL BAREQUET AND ALINA SHAIKHET, *Heilbronn's triangle problem*, In Proceedings of the twenty-third annual symposium on Computational geometry, pages 127–128, ACM, 2007.

[7] BORIS BUKH AND CHRISTOPHER COX, *Nearly orthogonal vectors and small antipodal spherical codes*.

[8] EMMANUEL J. CANDES, YONINA C. ELDAR, DEANNA NEEDELL, AND PAIGE RANDALL, *Compressed sensing with coherent and redundant dictionaries*, Applied and Computational Harmonic Analysis, **31** (1): 59–73, 2011.

[9] JAMES DEMMEL, MING GU, STANLEY EISENSTAT, IVAN SLAPNICAR, KREŠIMIR VESELIC, AND ZLATKO DRMAC, *Computing the singular value decomposition with high relative accuracy*, **299** (1–3): 21–80.

[10] PIER LUIGI DRAGOTTI AND MARTIN VETTERLI, *Wavelet footprints: theory, algorithms, and applications*, IEEE Transactions on Signal Processing, **51** (5): 1306–1323, 2003.

© ELEMENT, Zagreb
Paper OaM-18-17

[11] GÉRARD FAVIER, *Matrix and Tensor Decompositions in Signal Processing*, John Wiley & Sons, 2021.

[12] MATTHEW FICKUS, JOHN JASPER, DUSTIN G. MIXON, AND JESSE PETERSON, *Grouptheoretic constructions of erasure-robust frames*, Linear Algebra and its Applications, **479**: 131–154, 2015.

[13] MATTHEW FICKUS AND DUSTIN G. MIXON, *Numerically erasure-robust frames*, arXiv preprint arXiv:1202.4525, 2012.

[14] ADNAN GAVILI AND XIAO-PING ZHANG, *On the shift operator, graph frequency, and optimal filtering in graph signal processing*, IEEE Transactions on Signal Processing, **65** (23): 6303–6318, 2017.

[15] GENE H. GOLUB AND CHRISTIAN REINSCH, *Singular value decomposition and least squares solutions*, Numerische Mathematik, **14** (5): 403–420, 1970.

[16] GENEVIEVE GORRELL, *Generalized hebbian algorithm for incremental singular value decomposition in natural language processing*, In 11th conference of the European chapter of the association for computational linguistics, pages 97–104.

[17] VIVEK K. GOYAL, JELENA KOVACEVIC, AND JONATHAN A. KELNER, *Quantized frame expansions with erasures*, Applied and Computational Harmonic Analysis, **10** (3): 203–233, 2001.

[18] MOHAMMAD ALI HASANKHANI FARD AND SAEEDEH MOAZENI, *Signal reconstruction without phase by norm retrievable frames*, Linear and Multilinear Algebra, **69** (8): 1484–1499, 2021.

[19] JANOS KOMLOS, JANOS PINTZ, AND ENDRE SZEMEREDI, *On heilbronns triangle problem*, Journal of the London Mathematical Society, **24** (2): 385–396, 1981.

[20] SUN-YUAN KUNG, J. S. TAUR, AND M. Y. CHIU, *Application of svd networks to multiobject motion-shape analysis*, In Proceedings of IEEE Workshop on Neural Networks for Signal Processing, pages 413–422, IEEE, 1994.

[21] GITTA KUTYNIOK, ALI PEZESHKI, ROBERT CALDERBANK, AND TAOTAO LIU, *Robust dimension reduction, fusion frames, and grassmannian packings*, Applied and Computational Harmonic Analysis, **26** (1): 64–76, 2009.

[22] M. J. LAI AND Y. LIU, *The null space property for sparse recovery from multiple measurement vectors*, Applied and Computational Harmonic Analysis, **30** (3): 402–406, 2011.

[23] M. J. LAI AND Y. LIU, *The probabilistic estimates on the largest and smallest q-singular values of random matrices*, Mathematics of Computation, **84** (294): 1775–1794, 2015.

[24] HANNO LEFMANN, *On heilbronnâs problem in higher dimension*, Combinatorica, **23**: 669–680, 2003.

[25] JONG MIN LIM AND CHRISTOPHER L. DEMARCO, *Svd-based voltage stability assessment from phasor measurement unit data*, IEEE Transactions on Power Systems, **31** (4): 2557–2565, 2015.

[26] ALEXANDER E. LITVAK, ALAIN PAJOR, MARK RUDELSON, AND NICOLE TOMCZAK-JAEGERMANN, *Smallest singular value of random matrices and geometry of random polytopes*, Advances in Mathematics, **195** (2): 491–523, 2005.

[27] YANG LIU, *The probabilistic estimates of the largest strictly convex p-singular value of pregaussian random matrices*, Journal of Mathematics and Statistics, doi:10.3844/jmssp.2015, 2015.

[28] YANG LIU, *Probabilistic estimates of the largest strictly convex singular values of pregaussian random matrices*, Journal of Mathematics and Statistics, **11** (1): 7–15, 2015.

[29] YANG LIU, *Comparison on the robustness against erasure rates of numerically erasure-robust frames*, International Journal of Applied Mathematics, **33** (4): 585–590, 2020.

[30] YANG LIU AND YANG WANG, *On the decay of the smallest singular value of submatrices of rectangular matrices*, Asian-European Journal of Mathematics, **9** (04): 1650075, 2016.

[31] P.-A. LOF, T. SMED, G. ANDERSSON, AND D. J. HILL, *Fast calculation of a voltage stability index*, IEEE Transactions on Power Systems, **7** (1): 54–64, 1992.

[32] ROBERT MAHONY, TAREK HAMEL, AND JEAN-MICHEL PFLIMLIN, *Nonlinear complementary filters on the special orthogonal group*, IEEE Transactions on automatic control, **53** (5): 1203–1218, 2008.

[33] STÉPHANE MALLAT, *A wavelet tour of signal processing*, Academic press, 1999.

[34] SIDHARTH PRASAD MISHRA, UTTAM SARKAR, SUBHASH TARAPHDER, SANJAY DATTA, D. SWAIN, RESHMA SAIKHOM, SASMITA PANDA, AND MENALSH LAISHRAM, *Multivariate statistical data analysis-principal component analysis (pca)*, **7** (5): 60–78.

[35] BABACK MOGHADDAM, YAIR WEISS, AND SHAI AVIDAN, *Generalized spectral bounds for sparse lda*, In Proceedings of the 23rd international conference on Machine learning, pages 641–648, 2006.

[36] BRUCE MOORE, *Principal component analysis in linear systems: Controllability, observability, and model reduction*, IEEE transactions on automatic control, **26** (1): 17–32, 1981.

[37] DIJANA MOSIC, *The cmp inverse for rectangular matrices*, Aequationes mathematicae, **92** (4): 649–659, 2018.

[38] HENRIK OHLSSON, ALLEN YANG, ROY DONG, AND SHANKAR SASTRY, *Cprl-an extension of compressive sensing to the phase retrieval problem*, In Advances in Neural Information Processing Systems, pages 1367–1375, 2012.

[39] MARKUS PUSCHEL AND JELENA KOVACEVIC, *Real, tight frames with maximal robustness to erasures*, In Data Compression Conference, pages 63–72, IEEE, 2005.

[40] ROBERT QIU AND MICHAEL WICKS, *Cognitive Networked Sensing and Big Data*, Springer, 2013.

[41] RON RUBINSTEIN, TOMER PELEG, AND MICHAEL ELAD, *Analysis k-svd: A dictionarylearning algorithm for the analysis sparse model*, IEEE Transactions on Signal Processing, **61** (3): 661–677, 2012.

[42] MARK RUDELSON AND ROMAN VERSHYNIN, *Smallest singular value of a random rectangular matrix*, Communications on Pure and Applied Mathematics, **62** (12): 1707–1739, 2009.

[43] MARK RUDELSON AND ROMAN VERSHYNIN, *Non-asymptotic theory of random matrices: extreme singular values*, In Proceedings of the International Congress of Mathematicians, 2010.

[44] R. C. THOMPSON, *Singular value inequalities for matrix sums and minors*, Linear Algebra and its Applications, **11** (3): 251–269, 1975.

[45] ROBERT C THOMPSON, *Principal submatrices ix: Interlacing inequalities for singular values of submatrices*, Linear Algebra and its Applications, **5** (1): 1–12, 1972.

[46] ROMAN VERSHYNIN, *Introduction to the non-asymptotic analysis of random matrices*, arXiv preprint arXiv:1011.3027, 2010.

[47] HENGYOU WANG, WEN LI, LUJIN HU, CHANGLUN ZHANG, AND QIANG HE, *Structural smoothness low-rank matrix recovery via outlier estimation for image denoising*, Multimedia Systems, pages 1–15, 2021.

[48] BENEDIKT WEBER, PATRICK PAULTRE, AND JEAN PROULX, *Consistent regularization of nonlinear model updating for damage identification*, Mechanical Systems and Signal Processing, **23** (6): 1965–1985, 2009.

[49] ZHENBING ZENG AND LIANGYU CHEN, *Determining the heilbronn configuration of seven points in triangles via symbolic computation*, In International Workshop on Computer Algebra in Scientific Computing, pages 458–477, Springer, 2019.

[50] ANRU ZHANG AND RUNGANG HAN, *Optimal sparse singular value decomposition for high-dimensional high-order data*, Journal of the American Statistical Association, **114** (528): 1708–1725, 2019.

Operators and Matrices
www.ele-math.com
oam@ele-math.com