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ON THE ESTIMATION OF q–NUMERICAL RADIUS OF HILBERT SPACE OPERATORS

ARNAB PATRA AND FALGUNI ROY

Abstract. The objective of this article is to estimate the q-numerical radius of bounded linear operators on complex Hilbert spaces. One of our main results states that for a bounded linear operator T in a Hilbert space $\mathcal{H}$ and $q \in [0,1]$, the relation

$$\omega_q^2(T) \leqslant q^2\omega^2(T) + (1 - q^2 + q\sqrt{1-q^2})\|T\|^2$$

holds where $\omega(T)$, $\omega_q(T)$ are the numerical radius and q-numerical radius of T respectively. Several refined new upper bounds follow from this result. Finally, the q-numerical radius of 2×2 operator matrices is explored and several new results are established.

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