MULTIPLICATIVELY NUMERICAL RANGE–PRESERVING MAPS

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Abstract. Given a complex Hilbert space $\mathscr{H}$, we denote by $\mathscr{B}(\mathscr{H})$ the algebra of all bounded linear operators on $\mathscr{H}$, and $\mathscr{A}$, $\mathscr{B}$ two subsets of $\mathscr{B}(\mathscr{H})$ containing all operators of rank at most one. Let $W(A)$ the numerical range of $A \in \mathscr{B}(\mathscr{H})$. For an infinite-dimensional space $\mathscr{H}$, we prove that surjective maps $\phi_1, \phi_2 : \mathscr{A} \longrightarrow \mathscr{B}$ satisfy

$$W(\phi_1(A)\phi_2(B)) = W(AB), \quad (A, B \in \mathscr{A})$$

if and only if there exist $\mu, \nu \in \mathbb{C}$ with $\mu\nu = 1$, a bounded invertible linear operator U on $\mathscr{H}$ and a unitary operator V on $\mathscr{H}$ such that $\phi_1(A) = \mu VAU^{-1}$ and $\phi_2(A) = \nu UAV^*$ for all $A \in \mathscr{A}$. We also obtain an analogue result for the finite-dimensional case. Furthermore, some known results are obtained as immediate consequences of our main results.

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