

EUCLIDEAN OPERATOR RADIUS AND NUMERICAL RADIUS INEQUALITIES

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Abstract. Let T be a bounded linear operator on a complex Hilbert space \mathcal{H} . We obtain various lower and upper bounds for the numerical radius of T by developing the Euclidean operator radius bounds of a pair of operators, which are stronger than the existing ones. In particular, we develop an inequality that improves on the inequality

$$w(T) \geq \frac{1}{2}\|T\| + \frac{1}{4}\left|\|Re(T)\| - \frac{1}{2}\|T\|\right| + \frac{1}{4}\left|\|Im(T)\| - \frac{1}{2}\|T\|\right|.$$

Various equality conditions of the existing numerical radius inequalities are also provided. Further, we study the numerical radius inequalities of 2×2 off-diagonal operator matrices. Applying the numerical radius bounds of operator matrices, we develop upper bounds of $w(T)$ by using t -Aluthge transform. In particular, we improve the well known inequality

$$w(T) \leq \frac{1}{2}\|T\| + \frac{1}{2}w(\tilde{T}),$$

where $\tilde{T} = |T|^{1/2}U|T|^{1/2}$ is the Aluthge transform of T and $T = U|T|$ is the polar decomposition of T .

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