Operators and Matrices
Volume 19, Number 1 (2025), 49–67
doi:10.7153/oam-2025-19-04

MIXED PRODUCTS OF TOEPLITZ AND HANKEL OPERATORS ON THE FOCK–SOBOLEV SPACES

JUNMEI FAN, LIU LIU* AND YUFENG LU

Abstract. For entire functions f and g, we study the problem of when the mixed products of Toeplitz and Hankel operators $H_{\overline{f}}T_{\overline{g}}$ is bounded or compact on the Fock-Sobolev spaces $F^{p,m}(\mathbb{C}^n)$ with $1 \leqslant p < \infty$. This is a companion to Sarason's Toeplitz product problem which was completely solved for the Fock space by Cho-Park-Zhu in 2014. Our results here completely characterize the bounded and compact mixed product $H_{\overline{f}}T_{\overline{g}}$ on the Fock-Sobolev space.

Mathematics subject classification (2020): Primary 46A65, 47A15; Secondary 47B38.
Keywords and phrases: Toeplitz operators, Hankel operators, Fock-Sobolev space, boundedness.

REFERENCES

[1] A. ALEMAN, S. POTT AND C. REGUERA, *Sarason conjecture on the Bergman space*, Int. Math. Res. Not. Imrn. **2017**, 14 (2017), 4320–4349.
[2] C. BERGER AND L. COBURN, *Toeplitz operators and quantum mechanics*, J. Funct. Anal. **68**, 3 (1986), 273–299.
[3] C. BRIAN AND L. WICHARN, *Holomorphic Sobolev spaces and the generalized Segal-Bargman transform*, J. Funct. Anal. **217**, 1 (2004), 192–220.
[4] H. CHO, B. CHOE AND H. KOO, *Linear combinations of composition operators on the Fock-Sobolev spaces*, Potential. Anal. **41**, 4 (2014), 1223–1246.
[5] H. CHO, J. PARK AND K. ZHU, *Products of Toeplitz operators on the Fock space*, Proc. Amer. Math. Soc. **142**, 7 (2014), 2483–2489.
[6] H. CHO, K. ZHU, *Fock-Sobolev spaces and their Carleson measures*, J. Funct. Anal. **263**, 2483–2506 (2012).
[7] J. CHEN, X. WANG, J. XIA AND G. CAO, *Sarason's Toeplitz Product Problem on the Fock-Sobolev Space*, Acta Mathematica Sinica, English Series **34**, 2 (2018), 288–296.
[8] R. DOUGLAS, *Banach Algebra Techniques in Operator Theory* (second edition), Graduate Texts in Mathematics **179**, Springer, New York, 1998.
[9] L. HE AND G. CAO, *Fock-Sobolev spaces and weighted composition operators among them*, Commun. Math. Res. **32**, 4 (2016), 303–318.
[10] Z. HU AND X. LV, *Toeplitz operators on Fock spaces* $F^p(\varphi)$, Integr. Equ. Oper. Theory. **80**, 1 (2014), 33–59.
[11] J. ISRALOWITZ, *Compactness and essential norm properties of operators on generalized Fock spaces*, J. Operator. Theory. **73**, 2 (2015), 281–314.
[12] X. LV, *Hankel operators on Fock spaces* $F^p(\varphi)$, Complex Variables And Elliptic Equations **64**, 9 (2019), 1522–1533.
[13] P. MA, F. YAN, D. ZHENG AND K. ZHU, *Mixed products of Toeplitz and Hankel operators on the Fock space*, J. Operator Theory **84**, 1 (2020), 35–47.
[14] P. MA, F. YAN, D. ZHENG AND K. ZHU, *Products of Hankel operators on the Fock space*, J. Funct. Anal. **277**, 8 (2019), 2644–2663.
[15] F. NAZAROV, *A counterexample to Sarason's conjecture*, preprint, 1997.
[16] J. QIN AND X. WANG, *Mixed Product of Hankel and Toeplitz Operators on Fock-Sobolev Spaces*, Acta Math. Sin. (Engl. Ser.) **36**, 11 (2020), 1245–1255.

Paper OaM-19-04

[17] W. RUDIN, *Function Theory in the Unit Ball of* $\mathbb{C}^n$, Springer-Verlag, Berlin Heidelberg-New York, 1980.
[18] D. SARASON, *Products of Toeplitz operators*, in V. P. Havin, N. K. Nikolski (Eds.): Linear and Complex Analysis Problem Book 3, Part I, Lecture Notes in Math. Springer, Berlin, 1994, 318–319.
[19] H. SCHUSTER, H. BOMMIER-HATO, E. YOUSSFI AND K. ZHU, *Sarason's Toeplitz product problem for a class of Fock spaces*, Bull. Sci. Math. **141**, 5 (2017), 408–442.
[20] A. SCHUSTER AND D. VAROLIN, *Toeplitz Operators and Carleson Measures on Generalized Bargmann-Fock Spaces*, Integral Equations Operator Theory **72**, 3 (2012), 363–392.
[21] K. STROETHOFF AND D. ZHENG, *Bounded Toeplitz products on Bergman spaces of the unit ball*, J. Math. Anal. Appl. **325**, 1 (2007), 114–129.
[22] K. STROETHOFF AND D. ZHENG, *Products of Hankel and Toeplitz operators on the Bergman space*, J. Funct. Anal. **169**, 1 (1999), 289–313.
[23] X. WANG, G. CAO AND J. XIA, *Toeplitz Operators on Fock-Sobolev Spaces with positive measure symbols*, Science China Mathematics **57**, 7, 1443–1462.
[24] D. ZHENG, *The distribution function inequality and products of Toeplitz operators and Hankel operators*, J. Funct. Anal. **138**, 2 (1996), 477–501.
[25] D. ZHENG, *Toeplitz operators and Hankel operators on the Hardy space of the unit sphere*, J. Funct. Anal. **149**, 1 (1997), 1–24.
[26] K. ZHU, *Analysis on Fock Spaces*, Graduate Texts in Mathematics **263**, Springer, New York, 2012.