Operators and Matrices
Volume 19, Number 3 (2025), 321–337
doi:10.7153/oam-2025-19-21

THE BEREZIN RADIUS AND THE BEREZIN NORM ASSOCIATED WITH THE TENSOR PRODUCT

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Abstract. The Berezin range of a bounded operator A acting on a reproducing kernel Hilbert space $\mathscr{H}$ is the set $\mathbf{Ber}(A) := \{\langle A\hat{k}_\tau, \hat{k}_\tau\rangle : \tau \in \Theta\}$, where $\hat{k}_\tau$ is the normalized reproducing kernel for $\mathscr{H}$ at $\tau \in \Theta$. The Berezin radius (number) and the Berezin norms of an operator A are defined by $\mathbf{ber}(A) := \sup_{\tau\in\Theta} |\langle A\hat{k}_\tau, \hat{k}_\tau\rangle|$, $\|A\|_{\mathbf{ber},1} := \sup_{\tau,\mu\in\Theta} |\langle A\hat{k}_\tau, \hat{k}_\mu\rangle|$, and $\|A\|_{\mathbf{ber},2} := \sup_{\tau\in\Theta} \|A\hat{k}_\tau\|$ respectively. In this paper, we obtain some Berezin radius upper bounds for Hilbert space operators involving the tensor product. Moreover, the obtained upper bounds have been compared with the previously known bounds to demonstrate their reliability.

Mathematics subject classification (2020): 47A30, 47A63, 15A60.
Keywords and phrases: Berezin radius, Berezin norm, reproducing kernel, Cauchy-Schwarz inequality.

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