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SOME GENERALIZATIONS OF NUMERICAL RADII INEQUALITIES

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Abstract. This paper focuses on establishing new upper bounds for the numerical radius of operators on Hilbert spaces by utilizing the Moore-Penrose inverse and the generalized Cartesian decomposition. The obtained estimates enhance the existing body of knowledge and are systematically compared with results from the current literature. Our findings not only extend but also unify several recent contributions, offering a broader and more cohesive framework for understanding numerical radius inequalities. Through the application of the generalized Cartesian decomposition, we provide deeper insights into the behavior of numerical radii, building upon previous research and opening new directions for further investigation in this field.

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REFERENCES

[1] J. S. AUJLA, AND F. C. SILVA, *Weak majorization inequalities and convex functions*, Linear Algebra Appl. **369**, (2003) 217–233.
[2] M. AL-DOLAT, F. KITTANEH, *Upper bounds for the numerical radii of powers of Hilbert space operators*, Quaestiones Mathematicae **47** (2024), 341–352.
[3] S. BARIK, P. BHUNIA, AND K. PAUL, *A notion of the Cartesian decomposition and P-numerical radius bounds*, J. Anal. (2025), https://doi.org/10.1007/s41478-025-00938-1.
[4] P. BHUNIA, A. SEN, K. PAUL, *Generalized cartesian decomposition and numerical radius inequalities*, Rend. Circ. Mat. Palerm. **73** (2024), 887–897.
[5] R. BHATIA, F. KITTANEH, *Cartesian decompositions and Schatten norms*, Linear Algebra Appl. **318** (2000), 109–116.
[6] P. BHUNIA, F. KITTANEH, AND S. SAHOO, *Improved numerical radius bounds using the Moore–Penrose inverse*, Linear Algebra Appl. **711** (2025), 1–16.
[7] P. BHUNIA, S. JANA, AND K. PAUL, *Refined inequalities for the numerical radius of Hilbert space operators*, Rocky Mountain J. Math. **55** (2025), no. 2, 323–332.
[8] J. C. BOURIN, M. UCHIYAMA, *A matrix subadditivity inequality for $f(A+B)$ and $f(A)+f(B)$*, Linear Algebra Appl. **423** (2007), 512–518.
[9] D. CHOI, T. Y. TAM, P. ZHANG, *Extensions of Fischer's inequality*, Linear Algebra Appl. **569** (2019), 311–322.
[10] G. CORACH, A. MAESTRIPIERI, *Weighted generalized inverses, oblique projections, and least-squares problems*, Numer. Funct. Anal. Optim. **26** (6) (2005), 659–673.
[11] S. DRURY, *Principal powers of matrices with positive definite real part*, Linear Multilinear Algebra **63** (2015), 296–301.
[12] S. S. DRAGOMIR, *Power inequalities for the numerical radius of a product of two operators in Hilbert spaces*, Sarajevo J. Math. **18** (2009), 269–278.
[13] M. EL-HADDAD, F. KITTANEH, *Numerical radius inequalities for Hilbert space operators II*, Stud. Math. **182** (2007), 133–140.
[14] G. FONGI, M. GONZALEZ, *Moore-Penrose inverse and partial orders on Hilbert space operator*, Linear Algebra Appl. **674** (2023), 1–20.
[15] S. FURUICHI, H. R. MORADI, *Advances in mathematical inequality*, De Gruyter, 2020.

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[16] C. W. GROETSCH, *Generalized Inverses of Linear Operators*, Representation and Approximation, Marcel Dekker, New York, 1977.
[17] G. H. HARDY, J. E. LITTLEWOOD, G. PÓLYA, *Inequalities*, 2nd ed., Cambridge Univ. Press, Cambridge, 1988.
[18] Z. HEYDARBEYGI, M. SABABHEH, H. R. MORADI, *A Convex treatment of numerical radius inequalities*, Czechoslovak Math. J. **72** (2022), 601–614.
[19] A. GOURTY, M. A. IGHACHANE, AND F. KITTANEH, *Further A-numerical radius inequalities for semi-Hilbertian space operators*, J. Anal. **33**, 2023–2045 (2025).
[20] M. R. JABBARZADEH, V. KALEIBARY, *Inequalities for accretive-dissipative block matrices involving convex and concave functions*, Linear Multilinear Algebra **70** (2022), 395–410.
[21] F. KITTANEH, *Numerical radius inequalities for Hilbert space operators*, Stud. Math. **168** (2005), 73–80.
[22] F. KITTANEH, *A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix*, Stud. Math. **158** (2003), 11–17.
[23] F. KITTANEH, M. S. MOSLEHIAN, T. YAMAZAKI, *Cartesian decomposition and numerical radius inequalities*, Linear Algebra Appl. **471** (2015), 46–53.
[24] F. KITTANEH, M. SAKKIJHA, *Inequalities for accretive-dissipative matrices*, Linear Multilinear Algebra **67** (2019), 1037–1042.
[25] F. KITTANEH, *Notes on some inequalities for Hilbert space operators*, Publ. Res. Inst. Math. Sci. **24** (1988), 283–293.
[26] T. KOSEM, *Inequalities between* $||f(A+B)||$ *and* $||f(A)+f(B)||$, Linear Algebra Appl. **418** (2006), 153–160.
[27] B. MOND AND J. PEČARIĆ, *Remarks on Jensen's inequality for operator convex functions*, Ann. Univ. Mariae Curie-Sklodowska Sec. A, **47** (1993), 96–103.
[28] L. NASIRI, S. FURUICHI, *On a reverse of the Tan-Xie inequality for sector matrices and its applications*, J. Math. Inequal. **15** (2021), 1425–1434.
[29] M. SABABHEH, D. S. DJORDJEVĆ AND H. R. MORADI, *Numerical radius and rorm bounds via the Moore-Penrose Inverse*, Complex Anal. Oper. Theory (2024) **18**: 117, https://doi.org/10.1007/s11785-024-01560-y.
[30] A. SHEIKHHOSSEINI, M. KHOSRAVI, M. SABABHEH, *The weighted numerical radius*, Ann. Funct. Anal. **13** (2022), Paper No. 3, 15 pp.
[31] M. UCHIYAMA, *Subadditivity of eigenvalue sums*, Proc. Amer. Math. Soc. **134** (2006), 1405–1412.
[32] T. YAMAZAKI, *On upper and lower bounds of the numerical radius and an equality condition*, Stud. Math. **178** (2007), 83–89.