

## DETERMINANT FORMULAS FOR FINITE TOEPLITZ PLUS HANKEL MATRICES WITH RATIONAL SYMBOLS

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*Abstract.* In 1975 K. Michael Day produced an exact formula for the determinants of finite Toeplitz matrices whose symbols are rational. The answer is a sum that involves powers of the roots of the numerator of the symbol and whose coefficients depend on both the roots of the numerator and denominator. In this paper we prove an analogue of Day’s formula for determinants of finite Toeplitz plus Hankel matrices with rational symbols. The key to the proof is an exact formula for the finite determinants that involves a Fredholm determinant that can be explicitly computed. We apply the formula to find information about the limiting eigenvalues of the finite Toeplitz plus Hankel matrices.

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