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NUMERICAL RADIUS INEQUALITIES FOR OPERATOR MATRICES THROUGH MIXED SCHWARZ TYPE INEQUALITIES

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Abstract. We establish new numerical radius inequalities for $n \times n$ operator matrices based on mixed Schwarz type inequalities. Using polar decomposition $\mathscr{A}_{ij} = \mathscr{U}_{ij}|\mathscr{A}_{ij}|$ and continuous nonnegative functions $f, g : [0,\infty) \to [0,\infty)$, we develop a Cauchy Schwarz type numerical radius inequality for composite operator blocks of the form

$$\widetilde{\mathscr{T}_{ij}} = g\big(|\mathscr{B}_{ij}|\big)\, f\big(|\mathscr{A}_{ij}^*|\big)\, \mathscr{U}_{ij}, \qquad 1 \leqslant i, j \leqslant n.$$

This unified approach extends several classical inequalities for operator matrices and yields the estimate

$$w(\widetilde{\mathscr{T}}) \leqslant w\big(\widehat{\mathscr{T}^{(2)}}\big),$$

where $\widetilde{\mathscr{T}} = [\widetilde{\mathscr{T}_{ij}}]$ is an $n \times n$ operator matrix and $\widehat{\mathscr{T}^{(2)}}$ is an upper triangular scalar comparison matrix, whose entries depend only on the functions f and g.

As an application, choosing suitable functions f and g together with a specific choice of the block operator $\mathscr{B}_{ij}$ leads to a new inequality involving the Moore-Penrose inverse. The resulting estimates recover several known inequalities and also provide new sharper bounds for the numerical radius of operator matrices.

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