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SOME RESULTS ON WEIGHTED CRITICAL QUASILINEAR PROBLEMS

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Abstract. In this paper, we are concerned with a kind of quasilinear elliptic problems, which involves the Caffarelli-Kohn-Nirenberg inequality and critical exponents. By employing variational methods and analytical techniques, the existence of sign-changing solutions to the problem is proved.

1. Introduction

In this paper, we investigate the following elliptic problem:

$$\begin{cases} -\operatorname{div}\Big(\dfrac{|\nabla u|^{p-2}\nabla u}{|x|^{ap}}\Big) - \mu\dfrac{|u|^{p-2}u}{|x|^{p(a+1)}} = \dfrac{|u|^{p^*(a,b)-2}u}{|x|^{bp^*(a,b)}} + \lambda\dfrac{|u|^{q-2}u}{|x|^{dp^*(a,d)}}, \\ u \in W^{1,p}_{0,a}(\Omega), \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and

$$\begin{aligned} &0 \in \Omega,\ N \geqslant 3,\ \lambda > 0,\ 1 < p < N, \\ &0 \leqslant \mu < \bar{\mu},\ \bar{\mu} := \Big(\frac{N-p}{p} - a\Big)^p, \\ &0 \leqslant a < \frac{N-p}{p},\ a \leqslant b,\ d < a+1,\ p \leqslant q < p^*(a,d), \end{aligned}$$

where

$$p^*(a,b) := \frac{Np}{N-p(a+1-b)} \quad \text{and} \quad p^*(a,d) := \frac{Np}{N-p(a+1-d)}$$

are the critical Hardy-Sobolev exponents and $p^*(0,0) = p^* := Np/(N-p)$ is the critical Sobolev exponent. The space $W^{1,p}_{0,a}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm $\big(\int_\Omega |x|^{-ap}|\nabla u|^p\,dx\big)^{1/p}$.

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By a solution to the problem (1.1), we mean a function $u \in W^{1,p}_{0,a}(\Omega)$ such that the following equality holds for all $v \in W^{1,p}_{0,a}(\Omega)$:

$$\int_\Omega \Big(\frac{|\nabla u|^{p-2}\nabla u \nabla v}{|x|^{ap}} - \mu \frac{|u|^{p-2}uv}{|x|^{p(a+1)}} - \frac{|u|^{p^*(a,b)-2}uv}{|x|^{bp^*(a,b)}} - \lambda \frac{|u|^{q-2}uv}{|x|^{dp^*(a,d)}} \Big) \mathrm{d}x = 0.$$

By the standard elliptic regularity argument, $u \in C^1(\Omega \setminus \{0\})$.

Problem (1.1) is related to the Caffarelli-Kohn-Nirenberg inequality [3]:

$$\left(\int_{\mathbb{R}^N} \frac{|u|^{p^*(a,b)}}{|x|^{bp^*(a,b)}} \mathrm{d}x \right)^{\frac{p}{p^*(a,b)}} \leqslant C \int_{\mathbb{R}^N} \frac{|\nabla u|^p}{|x|^{ap}} \mathrm{d}x, \quad \forall\, u \in C_0^\infty(\mathbb{R}^N), \tag{1.2}$$

which is also named as the (weighted or general) Hardy-Sobolev inequality. For the sharp constants and extremal functions, see [9]. If $b = a+1$, then $p^*(a,b) = p$ and the following Hardy inequality holds [1] [16]:

$$\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{p(a+1)}} \mathrm{d}x \leqslant \frac{1}{\bar{\mu}} \int_{\mathbb{R}^N} \frac{|\nabla u|^p}{|x|^{ap}} \mathrm{d}x, \ \forall u \in C_0^\infty(\mathbb{R}^N), \tag{1.3}$$

where $\bar{\mu} = (\frac{N-p}{p} - a)^p$.

We employ the following norm in the space $W^{1,p}_{0,a}(\Omega)$ for $\mu < \bar{\mu}$:

$$||u|| = ||u||_{W^{1,p}_{0,a}(\Omega)} := \left(\int_\Omega \left(\frac{|\nabla u|^p}{|x|^{ap}} - \mu \frac{|u|^p}{|x|^{p(a+1)}} \right) \mathrm{d}x \right)^{\frac{1}{p}}. \tag{1.4}$$

By (1.3) it is equivalent to the norm $(\int_\Omega |x|^{-ap}|\nabla u|^p \mathrm{d}x)^{1/p}$ of the space $W^{1,p}_{0,a}(\Omega)$. According to the Hardy inequality, the following best constant is well-defined:

$$S_{\mu,a,b} := \inf_{u \in D^{1,p}_a(\mathbb{R}^N)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} \left(\frac{|\nabla u|^p}{|x|^{ap}} - \mu \frac{|u|^p}{|x|^{p(a+1)}} \right) \mathrm{d}x}{\left(\displaystyle\int_{\mathbb{R}^N} \frac{|u|^{p^*(a,b)}}{|x|^{bp^*(a,b)}} \mathrm{d}x \right)^{\frac{p}{p^*(a,b)}}},$$

where the space $D^{1,p}_a(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $(\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p \mathrm{d}x)^{1/p}$. By Lemma 2.1 of this paper, we can also define the following constant:

$$\Lambda_{\mu,a,d} := \inf_{u \in W^{1,p}_{0,a}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega \left(\frac{|\nabla u|^p}{|x|^{ap}} - \mu \frac{|u|^p}{|x|^{p(a+1)}} \right) \mathrm{d}x}{\displaystyle\int_\Omega \frac{|u|^p}{|x|^{dp^*(a,d)}} \mathrm{d}x}.$$

The elliptic problems involving the Hardy and Hardy-Sobolev inequalities have been studied by many authors, either in bounded domain or in the whole space $\mathbb{R}^N$, see [2]-[6], [8]-[10], [15]-[19] and the references therein. Many important results were

obtained in these publications and the results give us very good insight into these problems. In particular, when $p=2$, $a=0$ and $\mu<(\frac{N-2}{2})^2$, the problem (1.1) was investigated extensively. On the other hand, we know less about (1.1) when $p\neq 2$, $a\neq 0$ and $\mu<\bar{\mu}$. Thus it is meaningful for us to study (1.1) deeply.

The purpose of this paper is to investigate the existence of the sign-changing solutions to (1.1). In order to state clearly the conclusions of this paper, we need to explain some notations. $\alpha(\mu)$ and $\beta(\mu)$ $(\alpha(\mu)<\beta(\mu))$ are the zeroes of the function

$$f(t)=(p-1)t^p-\big(N-p(a+1)\big)t^{p-1}+\mu\,,\quad t\geqslant 0,\quad 0\leqslant\mu<\bar{\mu}\,.$$

The following constants are well defined and will be used in this paper:

$$\begin{aligned}
&\delta:=\tfrac{N-p}{p}-a,\\
&\tau_0:=\tfrac{N-p(a+1)}{N-p(a+1-b)},\\
&N':=p(a+1+\tfrac{p(a+1)-dp^*(a,d)}{p-1}),\\
&N'':=p(a+1+(p-1)(p(a+1)-dp^*(a,d))),\\
&q_1:=\max\big\{p,\ \tfrac{N-dp^*(a,d)-(p-1)(\beta(\mu)-\delta)}{\delta},\ \tfrac{N-dp^*(a,d)-(p-1)(\delta-\alpha(\mu)}{\delta}\big\},\\
&q_2:=\max\big\{p,\ \tfrac{N-dp^*(a,d)-(\beta(\mu)-\delta)}{\delta},\ \tfrac{N-dp^*(a,d)-(\delta-\alpha(\mu)}{\delta}\big\},\\
&\mu_1:=\big(\delta+\tfrac{p(a+1)-dp^*(a,d)}{p-1}\big)^{p-1}(\delta+dp^*(a,d)-p(a+1)),\\
&\mu_2:=\big(\delta-\tfrac{p(a+1)-dp^*(a,d)}{p-1}\big)^{p-1}(\delta+p(a+1)-dp^*(a,d)),\\
&\mu_3:=\big(\delta+p(a+1)-dp^*(a,d)\big)^{p-1}\big(\delta+(p-1)(dp^*(a,d)-p(a+1))\big),\\
&\mu_4:=\big(\delta-p(a+1)+dp^*(a,d)\big)^{p-1}\big(\delta+(p-1)(p(a+1)-dp^*(a,d))\big).
\end{aligned}$$

The main result of this paper is summarized in the following theorem, which is new when $0<a<(N-p)/p$ and $0<\mu<\bar{\mu}$. We can verify that the sets used in Theorem 1.1 for the parameters μ and q are not empty.

THEOREM 1.1. *Assume that one of the following conditions holds:*

(*i*) $1<p<2$, $0\leqslant\mu<\bar{\mu}$ *and* $q_1<q<p^*(a,d)$.

(*ii*) $2\leqslant p<N$, $0\leqslant\mu<\bar{\mu}$ *and* $q_2<q<p^*(a,d)$.

(*iii*) $1<p<2$, $q=p$, $0<\lambda<\Lambda_{\mu,a,d}$, $N>N'$ *and* $0\leqslant\mu<\min\{\mu_1,\mu_2\}$.

(*iv*) $2\leqslant p<N$, $q=p$, $0<\lambda<\Lambda_{\mu,a,d}$, $N>N''$ *and* $0\leqslant\mu<\min\{\mu_3,\mu_4\}$.

Then the problem (1.1) *has one pair of sign-changing solutions* $\pm u(x)$, *satisfying*

$$\int_\Omega\Big(\frac{|u|^{p^*(a,b)-p}}{|x|^{bp^*(a,b)}}+\lambda\frac{|u|^{q-p}}{|x|^{dp^*(a,d)}}\Big)\,v(u)^{p-1}u=0\,,$$

where $v(u)$ *is the first eigenfunction of the weighted eigenvalue problem*

$$\begin{cases}-\operatorname{div}\Big(\dfrac{|\nabla v|^{p-2}\nabla v}{|x|^{ap}}\Big)-\mu\dfrac{|v|^{p-2}v}{|x|^{p(a+1)}}=\gamma\Big(\dfrac{|u|^{p^*(a,b)-p}}{|x|^{bp^*(a,b)}}+\lambda\dfrac{|u|^{q-p}}{|x|^{dp^*(a,d)}}\Big)|v|^{p-2}v,\\[2ex] v\in W^{1,p}_{0,a}(\Omega).\end{cases}$$

REMARK 1.1. It is known that $S_{0,a,b}$ has the explicit minimizers ([9]):

$$V(x) = C\varepsilon^{a-\frac{N-p}{p}}\left(1+\left(\frac{|x|}{\varepsilon}\right)^{\frac{p(a+1)-bp^*(a,b)}{p-1}}\right)^{-\frac{N-p(a+1)}{p(a+1)-bp^*(a,b)}},$$

where $C>0$ is a particular constant and $\varepsilon>0$ is an arbitrary constant. On the other hand, when $0\leqslant a<\frac{N-p}{p}$ and $0\leqslant\mu<\bar{\mu}$, the extremals of $S_{\mu,a,b}$ and the existence and properties of positive solutions to (1.1) were investigated in [10] and [11].

This paper is organized as follows. In Section 2 some preliminary results are established. In Section 3 the asymptotic properties of the extremal functions related to $S_{\mu,a,b}$ are investigated. At last, we verify Theorem 1.1 in Section 4. In the following argument, $\eta=O(\varepsilon^{\tau})\,(\tau>0)$ means that there exists positive constant C such that $|\eta|\leqslant C\varepsilon^{\tau}$ for $\varepsilon>0$ small enough, $o(\varepsilon^t)$ means $|o(\varepsilon^t)|/\varepsilon^t\to 0$ as $\varepsilon\to 0$ and $o(1)$ stands for a generic infinitesimal value. We always denote the positive constants as C and omit $\mathrm{d}x$ in integrals for convenience.

2. Preliminary results

We summarize some required results.

LEMMA 2.1. *Suppose* $a\leqslant d<a+1$, $p\leqslant q\leqslant p^*(a,d)$, $0\leqslant\mu<\bar{\mu}$. *Then:*
(i) *there exists a constant* $C>0$ *such that*

$$\left(\int_\Omega\frac{|u|^q}{|x|^{dp^*(a,d)}}\right)^{p/q}\leqslant C\int_\Omega\left(\frac{|\nabla u|^p}{|x|^{ap}}-\mu\frac{|u|^p}{|x|^{p(a+1)}}\right),\quad\forall\,u\in W^{1,p}_{0,a}(\Omega);$$

(ii) *the embedding* $W^{1,p}_{0,a}(\Omega)\hookrightarrow L^q(\Omega,|x|^{-dp^*(a,d)})$ *is compact if* $p\leqslant q<p^*(a,d)$.

Proof. The statement (i) can be proved by employing the Hölder inequality, (1.2) and the equivalent norm (1.4) of $W^{1,p}_{0,a}(\Omega)$. The proof of (ii) can be found in [21].

LEMMA 2.2. ([10],[22]) *Let us suppose:*

$$0\leqslant a<\frac{N-p}{p},\ a\leqslant b<a+1,\ \textit{and}\ 0\leqslant\mu<\bar{\mu}.$$

Then the best constant $S_{\mu,a,b}$ *is achieved in* $\mathbb{R}^N$ *by the radial functions*

$$V_\varepsilon(x):=\varepsilon^{-\delta}U_{p,\mu}(\varepsilon^{-1}x)=\varepsilon^{-\delta}U_{p,\mu}(\varepsilon^{-1}|x|),\ \forall\,\varepsilon>0,$$

that satisfy

$$\int_{\mathbb{R}^N}\left(\frac{|\nabla V_\varepsilon(x)|^p}{|x|^{ap}}-\mu\frac{|V_\varepsilon(x)|^p}{|x|^{p(a+1)}}\right)=\int_{\mathbb{R}^N}\frac{|V_\varepsilon(x)|^{p^*(a,b)}}{|x|^{bp^*(a,b)}}=(S_{\mu,a,b})^{\frac{p^*(a,b)}{p^*(a,b)-p}}.$$

The function $U_{p,\mu}(x) = U_{p,\mu}(|x|)$ *is the unique radial solution of the limiting problem:*

$$\begin{cases} -\operatorname{div}\left(\dfrac{|\nabla u|^{p-2}\nabla u}{|x|^{ap}}\right) - \mu\dfrac{u^{p-1}}{|x|^{p(a+1)}} = \dfrac{u^{p^*(a,b)-1}}{|x|^{bp^*(a,b)}} \quad in \ \ \mathbb{R}^N\backslash\{0\}, \\ u \in D_a^{1,p}(\mathbb{R}^N), \qquad u > 0 \ \ in \ \ \mathbb{R}^N\backslash\{0\}, \end{cases} \tag{2.1}$$

satisfying

$$U_{p,\mu}(1) = \big(p^*(a,b)\,(\bar{\mu} - \mu)/p\big)^{\frac{1}{p^*(a,b)-p}}.$$

Furthermore, $U_{p,\mu}$ *have the following properties:*

$$\lim_{r\to 0} r^{\alpha(\mu)}\,U_{p,\mu}(r) = C_1, \ \ \lim_{r\to 0} r^{\alpha(\mu)+1}\,|U'_{p,\mu}(r)| = C_1\,\alpha(\mu),$$
$$\lim_{r\to +\infty} r^{\beta(\mu)}\,U_{p,\mu}(r) = C_2, \ \ \lim_{r\to +\infty} r^{\beta(\mu)+1}\,|U'_{p,\mu}(r)| = C_2\,\beta(\mu),$$

where $C_i (i = 1,2)$ *are positive constants and* $\alpha(\mu)$ *and* $\beta(\mu)$ *are zeroes of the function*

$$f(t) = (p-1)t^p - \big(N - p(a+1)\big)t^{p-1} + \mu\,, \quad t \geqslant 0,$$

that satisfy

$$0 \leqslant \alpha(\mu) < \frac{N - p(a+1)}{p} < \beta(\mu) < \frac{N - p(a+1)}{p-1}. \tag{2.2}$$

Furthermore, there exist positive constants $C_3 = C_3(\mu,p,a,b)$ *and* $C_4 = C_4(\mu,p,a,b)$ *such that*

$$C_3 \leqslant U_{p,\mu}(x)\big(|x|^{\frac{\alpha(\mu)}{\delta}} + |x|^{\frac{\beta(\mu)}{\delta}}\big)^{\delta} \leqslant C_4.$$

We mention that the properties of positive solutions to (1.1) were investigated in a recent paper [11] and the following results are already known.

LEMMA 2.3. ([11]) *Suppose* $1 < p < N$ *and* $0 < \mu < \bar{\mu}$. *Assume that* $u \in W^{1,p}_{0,a}(\Omega)$ *is a positive solution to the problem* (1.1). *Then:*

(i) *there exists some constants* $\rho > 0$ *small and* $C > 0$, *such that*

$$u(x) \geqslant C\,|x|^{-\alpha(\mu)}, \quad \forall x \in B_\rho(0)\setminus\{0\};$$

(ii) $u \in L^r(\Omega, |x|^{-bp^*(a,b)})$, $\forall r \in \big(1, \frac{N\tau_0}{\alpha(\mu)}\big)$;

(iii) $|x|^{-a}\,|\nabla u| \in L^r(\Omega, |x|^{-bp^*(a,b)})$, $r \in \big(1, \frac{N\tau_0}{\alpha(\mu)+a+1}\big)$.

3. Asymptotic property of the extremal function

Let $V_\varepsilon(x)$ be the functions in Lemma 2.2. Take $\rho > 0$ small enough such that $B_\rho(0) \subset \Omega$, $\varphi(x) \in C_0^\infty(\Omega)$, $0 \leqslant \varphi(x) \leqslant 1$, $\varphi(x) = 1$ for $|x| \leqslant \frac{\rho}{2}$ and $\varphi(x) = 0$ for $|x| \geqslant \rho$. Setting $u_\varepsilon(x) = \varphi(x)V_\varepsilon(x)$, we have the following estimates.

LEMMA 3.1. ([12]) *As $\varepsilon \to 0$ we have:*

$$\int_{\mathbb{R}^N}\Big(\frac{|\nabla u_\varepsilon|^p}{|x|^{ap}} - \mu\frac{u_\varepsilon^p}{|x|^{p(a+1)}}\Big) = (S_{\mu,a,b})^{\frac{p^*(a,b)}{p^*(a,b)-p}} + O(\varepsilon^{\beta(\mu)p+p(a+1)-N}),$$

$$\int_\Omega \frac{u_\varepsilon^{p^*(a,b)}}{|x|^{bp^*(a,b)}} = (S_{\mu,a,b})^{\frac{p^*(a,b)}{p^*(a,b)-p}} + O(\varepsilon^{(\beta(\mu)+b)p^*(a,b)-N}),$$

$$\int_\Omega \frac{u_\varepsilon^q}{|x|^{dp^*(a,d)}} \geqslant \begin{cases} C\varepsilon^{N-dp^*(a,d)-q\delta}, & \frac{N-dp^*(a,d)}{\beta(\mu)} < q < p^*(a,d), \\ C\varepsilon^{q(\beta(\mu)-\delta)}|\ln\varepsilon|, & q = \frac{N-dp^*(a,d)}{\beta(\mu)}, \\ C\varepsilon^{q(\beta(\mu)-\delta)}, & 1 \leqslant q < \frac{N-dp^*(a,d)}{\beta(\mu)}, \end{cases}$$

$$\int_\Omega \frac{u_\varepsilon^q}{|x|^{dp^*(a,d)}} \to 0, \quad 1 \leqslant q < p^*(a,d).$$

LEMMA 3.2. *Suppose $0 < \mu < \bar{\mu}$ and $0 < q \leqslant p^*(a,b) - 1$. Assume that $u \in W_{0,a}^{1,p}(\Omega)$ is a positive solution of the problem (1.1). Then as $\varepsilon \to 0$ we have*

$$\int_\Omega \frac{|\nabla u||\nabla u_\varepsilon|^{p-1}}{|x|^{ap}} = \begin{cases} O(\varepsilon^{p-1+\delta-\alpha(\mu)}), & \alpha(\mu)+(p-1)\beta(\mu) > p\delta, \\ O(\varepsilon^{p-1+\delta-\alpha(\mu)}|\ln\varepsilon|), & \alpha(\mu)+(p-1)\beta(\mu) = p\delta, \\ O(\varepsilon^{(p-1)(\beta(\mu)-\delta+1)}), & \alpha(\mu)+(p-1)\beta(\mu) < p\delta, \end{cases} \tag{3.1}$$

$$\int_\Omega \frac{|\nabla u|^{p-1}|\nabla u_\varepsilon|}{|x|^{ap}} = \begin{cases} O(\varepsilon^{1+(p-1)(\delta-\alpha(\mu))}), & (p-1)\alpha(\mu)+\beta(\mu) > p\delta, \\ O(\varepsilon^{1+\beta(\mu)-\delta}|\ln\varepsilon|), & (p-1)\alpha(\mu)+\beta(\mu) = p\delta, \\ O(\varepsilon^{1+\beta(\mu)-\delta}), & (p-1)\alpha(\mu)+\beta(\mu) < p\delta, \end{cases} \tag{3.2}$$

$$\int_\Omega \frac{uu_\varepsilon^q}{|x|^{bp^*(a,b)}} = \begin{cases} O(\varepsilon^{N\tau_0-q\delta-\alpha(\mu)}), & \alpha(\mu)+q\beta(\mu) > N\tau_0, \\ O(\varepsilon^{q(\beta(\mu)-\delta)}|\ln\varepsilon|), & \alpha(\mu)+q\beta(\mu) = N\tau_0, \\ O(\varepsilon^{q(\beta(\mu)-\delta)}), & \alpha(\mu)+q\beta(\mu) < N\tau_0, \end{cases} \tag{3.3}$$

$$\int_\Omega \frac{u^q u_\varepsilon}{|x|^{bp^*(a,b)}} = \begin{cases} O(\varepsilon^{N\tau_0-\delta-q\alpha(\mu)}), & q\alpha(\mu)+\beta(\mu) > N\tau_0, \\ O(\varepsilon^{\beta(\mu)-\delta}|\ln\varepsilon|), & q\alpha(\mu)+\beta(\mu) = N\tau_0, \\ O(\varepsilon^{\beta(\mu)-\delta}), & q\alpha(\mu)+\beta(\mu) < N\tau_0, \end{cases} \tag{3.4}$$

$$\int_\Omega \frac{uu_\varepsilon^{p-1}}{|x|^{p(a+1)}} = \begin{cases} O(\varepsilon^{\delta-\alpha(\mu)}), & \alpha(\mu)+(p-1)\beta(\mu) > p\delta, \\ O(\varepsilon^{\delta-\alpha(\mu)}|\ln\varepsilon|), & \alpha(\mu)+(p-1)\beta(\mu) = p\delta, \\ O(\varepsilon^{(p-1)(\beta(\mu)-\delta)}), & \alpha(\mu)+(p-1)\beta(\mu) < p\delta, \end{cases} \tag{3.5}$$

$$\int_\Omega \frac{u^{p-1}u_\varepsilon}{|x|^{p(a+1)}} = \begin{cases} O(\varepsilon^{(p-1)(\delta-\alpha(\mu))}), & (p-1)\alpha(\mu)+\beta(\mu) > p\delta, \\ O(\varepsilon^{\beta(\mu)-\delta}|\ln\varepsilon|), & (p-1)\alpha(\mu)+\beta(\mu) = p\delta, \\ O(\varepsilon^{\beta(\mu)-\delta}), & (p-1)\alpha(\mu)+\beta(\mu) < p\delta. \end{cases} \tag{3.6}$$

Proof. The statements (3.1)-(3.6) can be verified by the Hölder inequality, Lemma 2.3 and Lemma 3.1. For simplicity we only prove (3.1). Note that the following equality is useful:

$$N\tau_0 + bp^*(a,b) = N. \tag{3.7}$$

VERIFICATION OF (3.1). Assume that $\alpha(\mu)+(p-1)\beta(\mu) > p\delta$. By taking $\tau > 0$ small we deduce that

$$N + \frac{bp^*(a,b)}{\frac{N\tau_0}{\alpha(\mu)+a+1} - \tau - 1} < ((a+1)(p-1)+\beta(\mu)(p-1))\frac{\frac{N\tau_0}{\alpha(\mu)+a+1} - \tau}{\frac{N\tau_0}{\alpha(\mu)+a+1} - \tau - 1}.$$

Consequently,

$$\begin{aligned}
&\int_\Omega \frac{|\nabla u||\nabla u_\varepsilon|^{p-1}}{|x|^{ap}} \\
&\leqslant \left(\int_\Omega \frac{(|x|^{-a}|\nabla u|)^{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau}}{|x|^{bp^*(a,b)}}\right)^{\frac{1}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau}} \\
&\quad\times \left(\int_\Omega |x|^{\frac{bp^*(a,b)}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau-1}}\left(|x|^{-a(p-1)}|\nabla u_\varepsilon|^{p-1}\right)^{\frac{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau-1}}\right)^{\frac{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau-1}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau}} \\
&\leqslant C\varepsilon^{-(\frac{N-p}{p})(p-1)}\left(\int_0^{\frac{\rho}{\varepsilon}} \varepsilon^{\frac{bp^*(a,b)}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau-1}+N} r^{\frac{bp^*(a,b)}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau-1}+N-1}\right. \\
&\quad\left.\times \left(r^{-(a+1)(p-1)}\left(r^{\frac{\alpha(\mu)}{\delta}}+r^{\frac{\beta(\mu)}{\delta}}\right)^{-\delta(p-1)}\right)^{\frac{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau-1}} dr\right)^{\frac{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau-1}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau}} \\
&\leqslant C\varepsilon^{-(\frac{N-p}{p})(p-1)}\left(\int_0^1 + \int_1^{\frac{\rho}{\varepsilon}}\right)^{\frac{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau-1}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau}} \\
&\leqslant C\varepsilon^{-(\frac{N-p}{p})(p-1)}\left(O\left(\varepsilon^{\frac{bp^*(a,b)}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau-1}+N}\right)+O\left(\varepsilon^{\frac{bp^*(a,b)}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau-1}+N}\right)\right)^{\frac{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau-1}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau}}
\end{aligned}$$

$$\leqslant C\varepsilon^{-(\frac{N-p}{p})(p-1)+N-(\alpha(\mu)+a+1)-\frac{(\alpha(\mu)+a+1)\tau}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau}}$$
$$\leqslant C\varepsilon^{(p-1)+(\delta-\alpha(\mu))-\frac{(\alpha(\mu)+a+1)\tau}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau}}.$$

Since τ is arbitrary, taking the limit as $\tau\to 0$ we have

$$\int_\Omega \frac{|\nabla u||\nabla u_\varepsilon|^{p-1}}{|x|^{ap}} \leqslant C\varepsilon^{(p-1)+(\delta-\alpha(\mu))}.$$

Assume that $\alpha(\mu)+(p-1)\beta(\mu)<p\delta$. By taking $\tau>0$ small we deduce that

$$N+\frac{bp^*(a,b)}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau-1} > ((a+1)(p-1)+\beta(\mu)(p-1))\frac{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau-1}.$$

By direct calculation we have

$$\begin{aligned}
&\int_\Omega \frac{|\nabla u||\nabla u_\varepsilon|^{p-1}}{|x|^{ap}}\\
&\leqslant \left(\int_\Omega \frac{(|x|^{-a}|\nabla u|)^{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau}}{|x|^{bp^*(a,b)}}\right)^{\frac{1}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau}}\\
&\quad\times\left(\int_\Omega |x|^{\frac{bp^*(a,b)}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau-1}}\left(|x|^{-a(p-1)}|\nabla u_\varepsilon|^{p-1}\right)^{\frac{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau-1}}\right)^{\frac{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau-1}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau}}\\
&\leqslant C\varepsilon^{-(\frac{N-p}{p})(p-1)}\left(\int_0^{\frac{\rho}{\varepsilon}} \varepsilon^{\frac{bp^*(a,b)}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau-1}+N} r^{\frac{bp^*(a,b)}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau-1}+N-1}\right.\\
&\quad\left.\times\left(r^{-(a+1)(p-1)}\left(r^{\frac{\alpha(\mu)}{\delta}}+r^{\frac{\beta(\mu)}{\delta}}\right)^{-\delta(p-1)}\right)^{\frac{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau-1}} dr\right)^{\frac{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau-1}{\frac{N\tau_0}{\alpha(\mu)+a+1}-\tau}}\\
&\leqslant C\varepsilon^{-(\frac{N-p}{p})(p-1)+(a+1)(p-1)+\beta(\mu)(p-1)}\\
&= O(\varepsilon^{(p-1)(\beta(\mu)-\delta+1)}).
\end{aligned}$$

If $\alpha(\mu)+(p-1)\beta(\mu)=p\delta$, by repeating the above argument we have

$$\int_\Omega \frac{|\nabla u||\nabla u_\varepsilon|^{p-1}}{|x|^{ap}} = O(\varepsilon^{(p-1)(\beta(\mu)-\delta+1)}|\ln\varepsilon|).$$

The proof of this lemma is thus completed.

4. Existence of sign-changing solutions

In this section, we investigate the sign-changing solutions to the problem (1.1). We have to overcome the singularity of the positive solutions to (1.1).

For $\nu \geqslant 0$ small and $u \in W^{1,p}_{0,a}(\Omega)$, we define

$$J_\nu(u) = \frac{1}{p}\|u\|^p - \frac{1}{p^*(a,b)-\nu}\int_\Omega \frac{|u|^{p^*(a,b)-\nu}}{|x|^{bp^*(a,b)}} - \frac{\lambda}{q}\int_\Omega \frac{|u|^q}{|x|^{dp^*(a,d)}}, \tag{4.1}$$

$$\Lambda_\nu = \{u \in W^{1,p}_{0,a}(\Omega)\,;\, \langle J'_\nu(u), u\rangle = 0\,,\, u \not\equiv 0\}, \tag{4.2}$$

$$c_{1,\nu} = \inf_{u\in\Lambda_\nu} J_\nu(u)\,. \tag{4.3}$$

Then $J_\nu \in C^1(W^{1,p}_{0,a}(\Omega), \mathbb{R})$. Moreover, for $\nu' > 0$ small enough, there exists $\alpha_0 > 0$ such that the following lower bound holds:

$$c_{1,\nu} \geqslant \alpha_0, \quad \forall \nu \in [0, \nu'].$$

We recall the following existence result related to the positive solutions of (1.1).

LEMMA 4.1. ([10]) *Suppose* $N \geqslant 3$, $\lambda > 0$, $a \leqslant b, d < a+1$, $0 \leqslant \mu < \bar{\mu}$. *Assume that one of the following conditions holds:*

(i) $q = p$, $0 < \lambda < \Lambda_{\mu,a,d}$, $N \geqslant p^2(a+1) + (1-p)dp^*(a,d)$ *and*

$$0 \leqslant \mu < \widetilde{\mu} := \frac{N - (p^2(a+1) + (1-p)dp^*(a,d))}{p}\left(\frac{N - dp^*(a,d)}{p}\right)^{p-1}.$$

(ii) $\lambda > 0$, $\widetilde{q} < q < p^*(a,d)$, *where*

$$\widetilde{q} := \max\left\{p\,,\ \frac{N - dp^*(a,d)}{\beta(\mu)}\,,\ \frac{p(2N - dp^*(a,d) - p(a+1+\beta(\mu)))}{N - p(a+1)}\right\}.$$

Then the problem (1.1) *has a mountain-pass-type positive solution* $u_1 \in \Lambda_0$.

It should be mentioned that the solution u_1 has the following property [20]:

$$J_0(u_1) = \sup_{t\in\mathbb{R}} J_0(t\,u_1) = c_{1,0} := \inf_{u\in\Lambda_0} J_0(u).$$

LEMMA 4.2. ([10]) *For* $\varepsilon > 0$ *small enough, there exists a constant* $C > 0$ *such that*

$$\sup_{t\geqslant 0} J_0(tu_\varepsilon) \leqslant \left(\frac{1}{p} - \frac{1}{p^*(a,b)}\right)(S_{\mu,a,b})^{\frac{p^*(a,b)}{p^*(a,b)-p}} - C\int_\Omega \frac{u_\varepsilon^q}{|x|^{dp^*(a,d)}} + O\big(\varepsilon^{p(\beta(\mu)-\delta)}\big).$$

To obtain the sign-changing solutions, we employ the min-max principle ([7]). To this end, let $B \subset W^{1,p}_{0,a}(\Omega)$ be a closed symmetric set. Then the Krasnoselski genus $i(B)$ is well defined. Fix $\rho > 0$ and define

$$S_\rho = \big\{u | u \in W^{1,p}_{0,a}(\Omega),\ \|u\| = \rho\,\big\},$$

$$\mathscr{H} = \big\{h | h : W^{1,p}_{0,a}(\Omega) \to W^{1,p}_{0,a}(\Omega) \text{ is an odd homeomorphism}\big\},$$

$$\mathscr{F}_2 = \big\{B | B \subset W^{1,p}_{0,a}(\Omega) \text{ is closed symmetric}\,,\ i(h(B) \cap S_\rho\,) \geqslant 2\,, \forall h \in \mathscr{H}\,\big\}.$$

The following result in the sub-critical case can be obtained by the min-max principle ([7]) and the proof is omitted for simplicity.

LEMMA 4.3. *There is a* $\nu^* > 0$, *such that for every* $\nu \in (0, \nu^*)$, *the problem*

$$\begin{cases} -\operatorname{div}\Big(\dfrac{|\nabla u|^{p-2}\nabla u}{|x|^{ap}}\Big) - \mu\dfrac{|u|^{p-2}u}{|x|^{p(a+1)}} = \dfrac{|u|^{p^*(a,b)-2-\nu}u}{|x|^{bp^*(a,b)}} + \lambda\dfrac{|u|^{q-2}u}{|x|^{dp^*(a,d)}}, \\ u \in W^{1,p}_{0,a}(\Omega), \end{cases} \tag{4.4}$$

has a pair of sign-changing solutions $\pm u_{2,\nu}$ *satisfying*

$$\int_\Omega \Big(\frac{|u_{2,\nu}|^{p^*(a,b)-p-\nu}}{|x|^{bp^*(a,b)}} + \lambda\frac{|u_{2,\nu}|^{q-p}}{|x|^{dp^*(a,d)}} \Big)\, v(u_{2,\nu})^{p-1}u_{2,\nu} = 0,$$

where $v(u_{2,\nu})$ *is the first eigenfunction of the weighted eigenvalue problem*

$$\begin{cases} -\operatorname{div}\Big(\dfrac{|\nabla v|^{p-2}\nabla v}{|x|^{ap}}\Big) - \mu\dfrac{|v|^{p-2}v}{|x|^{p(a+1)}} = \gamma\Big(\dfrac{|u_{2,\nu}|^{p^*(a,b)-p-\nu}}{|x|^{bp^*(a,b)}} + \lambda\dfrac{|u_{2,\nu}|^{q-p}}{|x|^{dp^*(a,d)}}\Big)|v|^{p-2}v, \\ v \in W^{1,p}_{0,a}(\Omega). \end{cases}$$

Furthermore,

$$c_{2,\nu} := \inf_{A\in\mathscr{F}_2}\sup_{w\in A} J_\nu(w) = J_\nu(u_{2,\nu}).$$

Note that the sets in Theorem 1.1 for the parameters q and μ are smaller than those in Lemma 4.1 respectively. Thus under the assumptions of Theorem 1.1 we can get a positive solution u_1 to (1.1). Furthermore, we have the following estimate for the sub-critical problem (4.4).

LEMMA 4.4. *Under the assumptions of Theorem 1.1, there exist* $\sigma > 0$ *and* $\nu^{**} > 0$ *such that*

$$c_{2,\nu} \leqslant c_{1,\nu} + \Big(\frac{1}{p} - \frac{1}{p^*(a,b)}\Big)\big(S_{\mu,a,b}\big)^{\frac{p^*(a,b)}{p^*(a,b)-p}} - \sigma, \ \ \forall\, \nu \in (0, \nu^{**}). \tag{4.5}$$

Proof. Arguing as in [8] and [18], we have that $c_{1,\nu} \to c_{1,0}$ and $c_{2,\nu} \to c_{2,0}$ as $\nu \to 0$.

The following elementary inequalities are well known: $\forall\, r \in [1, +\infty)$, there exists a constant $C = C(r) > 0$ such that

$$|A+B|^r \leqslant |A|^r + B|^r + C\big(|A|^{r-1}|B| + |A|\,|B|^{r-1}\big), \quad \forall A,\, B \in \mathbb{R},$$
$$|A+B|^r \geqslant |A|^r + |B|^r - C\big(|A|^{r-1}|B| + |A|\,|B|^{r-1}\big), \quad \forall A,\, B \in \mathbb{R}.$$

Set $\Gamma_\varepsilon = \operatorname{span}\{u_\varepsilon, u_1\}$, where u_ε and u_1 are the functions defined as in Lemmas 3.1 and 4.1. Then $\Gamma_\varepsilon \in \mathscr{F}_2$ and

$$c_{2,\nu} \leqslant \sup_{w\in\Gamma_\varepsilon} J_\nu(w) = \sup_{A,B\in\mathbb{R}} J_\nu(Au_1 + Bu_\varepsilon).$$

Consequently,

$$
\begin{aligned}
&J_\nu(Au_1+Bu_\varepsilon)\\
&\quad=\frac{1}{p}\|Au_1+Bu_\varepsilon\|^p-\frac{\lambda}{q}\int_\Omega\frac{|Au_1+Bu_\varepsilon|^q}{|x|^{dp^*(a,d)}}-\frac{1}{p^*(a,b)-\nu}\int_\Omega\frac{|Au_1+Bu_\varepsilon|^{p^*(a,b)-\nu}}{|x|^{bp^*(a,b)}}\\
&\quad\leqslant J_\nu(Au_1)+J_\nu(Bu_\varepsilon)\\
&\qquad+\ C|A|^{p-1}|B|\int_\Omega|x|^{-ap}|\nabla u_1|^{p-1}|\nabla u_\varepsilon|+C|A||B|^{p-1}\int_\Omega|x|^{-ap}|\nabla u_1||\nabla u_\varepsilon|^{p-1}\\
&\qquad+C|A|^{p^*(a,b)-1-\nu}|B|\int_\Omega\frac{u_1^{p^*(a,b)-1-\nu}u_\varepsilon}{|x|^{bp^*(a,b)}}+C|A||B|^{p^*(a,b)-1-\nu}\int_\Omega\frac{u_1u_\varepsilon^{p^*(a,b)-1-\nu}}{|x|^{bp^*(a,b)}}\\
&\qquad+C|A|^{p-1}|B|\int_\Omega\frac{u_1^{p-1}u_\varepsilon}{|x|^{p(a+1)}}\ +\ C|A||B|^{p-1}\int_\Omega\frac{u_1u_\varepsilon^{p-1}}{|x|^{p(a+1)}}\\
&\qquad+C|A|^{q-1}|B|\int_\Omega\frac{u_1^{q-1}u_\varepsilon}{|x|^{dp^*(a,d)}}\ +\ C|A||B|^{q-1}\int_\Omega\frac{u_1u_\varepsilon^{q-1}}{|x|^{dp^*(a,d)}}\\
&\quad\leqslant J_\nu(Au_1)+J_\nu(Bu_\varepsilon)\\
&\qquad+C(|A|^p+|B|^p)\,(\varepsilon^{\beta(\mu)-\delta}+\varepsilon^{(p-1)(\beta(\mu)-\delta)}+\varepsilon^{\alpha(\mu)-\delta}+\varepsilon^{(p-1)(\alpha(\mu)-\delta)})|\ln\varepsilon|\\
&\qquad+C\big(|A|^{p^*(a,b)-\nu}+|B|^{p^*(a,b)-\nu}\big)\,(\varepsilon^{\beta(\mu)-\delta}+\varepsilon^{\delta-\alpha(\mu)})|\ln\varepsilon|\\
&\qquad+C(|A|^q+|B|^q)\,(\varepsilon^{\beta(\mu)-\delta}+\varepsilon^{\delta-\alpha(\mu)})|\ln\varepsilon|\,.
\end{aligned}
$$

By the above estimates we get that

$$\lim_{A,B\to\infty}J_\nu(Au_1+Bu_\varepsilon)=-\infty\ \text{ for small enough }\ \varepsilon>0\,.$$

Therefore we may assume that A and B are in bounded sets. From Lemmas 3.1, 3.2 and 4.3 it follows that

$$
\begin{aligned}
&J_\nu(Au_1+Bu_\varepsilon)\\
&\quad\leqslant J_\nu(Au_1)+J_\nu(Bu_\varepsilon)\\
&\qquad\quad+C\big(\varepsilon^{(p-1)(\beta(\mu)-\delta)}+\varepsilon^{\beta(\mu)-\delta}+\varepsilon^{(p-1)(\delta-\alpha(\mu))}+\varepsilon^{\delta-\alpha(\mu)}\big)|\ln\varepsilon|\\
&\quad\leqslant c_{1,\nu}+J_0(Bu_\varepsilon)\\
&\qquad\quad+C\big(\varepsilon^{(p-1)(\beta(\mu)-\delta)}+\varepsilon^{\beta(\mu)-\delta}+\varepsilon^{(p-1)(\delta-\alpha(\mu))}+\varepsilon^{\delta-\alpha(\mu)}\big)|\ln\varepsilon|+I_1\\
&\quad\leqslant c_{1,\nu}+\Big(\frac{1}{p}-\frac{1}{p^*(a,b)}\Big)\big(S_{\mu,a,b}\big)^{\frac{p^*(a,b)}{p^*(a,b)-p}}-C\int_\Omega\frac{u_\varepsilon^q}{|x|^{dp^*(a,d)}}\\
&\qquad\quad+\ C\big(\varepsilon^{(p-1)(\beta(\mu)-\delta)}+\varepsilon^{\beta(\mu)-\delta}+\varepsilon^{(p-1)(\delta-\alpha(\mu))}+\varepsilon^{\delta-\alpha(\mu)}\big)|\ln\varepsilon|+I_1,
\end{aligned}
$$

where

$$I_1:=\frac{|B|^{p^*(a,b)}}{p^*(a,b)}\int_\Omega\frac{u_\varepsilon^{p^*(a,b)}}{|x|^{bp^*(a,b)}}\ -\ \frac{|B|^{p^*(a,b)-\nu}}{p^*(a,b)-\nu}\int_\Omega\frac{u_\varepsilon^{p^*(a,b)-\nu}}{|x|^{bp^*(a,b)}}\ .$$

(i). Assume that $1<p<2$ and $q>q_1$, where

$$q_1=\max\{p,\ \tfrac{N-dp^*(a,d)-(p-1)(\beta(\mu)-\delta)}{\delta},\ \tfrac{N-dp^*(a,d)-(p-1)(\delta-\alpha(\mu)}{\delta}\}.$$

Then

$$\begin{aligned}&N-dp^*(a,d)-q\delta<(p-1)(\beta(\mu)-\delta)<\beta(\mu)-\delta,\\&N-dp^*(a,d)-q\delta<(p-1)(\delta-\alpha(\mu))<\delta-\alpha(\mu).\end{aligned}$$

Since

$$\frac{N-dp^*(a,d)-(p-1)(\beta(\mu)-\delta)}{\delta}>\frac{N-dp^*(a,d)}{\beta(\mu)},$$

from Lemma 3.1 it follows that there exists a constant $\sigma>0$ such that

$$\begin{aligned}C\big(\varepsilon^{(p-1)(\beta(\mu)-\delta)}+\varepsilon^{\beta(\mu)-\delta}+\varepsilon^{(p-1)(\delta-\alpha(\mu))}+\varepsilon^{\delta-\alpha(\mu)}\big)|\ln\varepsilon|&\\ -C\int_\Omega\frac{u_\varepsilon^q}{|x|^{dp^*(a,d)}}&\leqslant-2\sigma.\end{aligned}\tag{4.6}$$

Choose $\nu^{**}>0$ small enough such that $I_1<\sigma$ for $0<\nu<\nu^{**}$. Then

$$c_{2,\nu}\leqslant J_\nu(Au_1+Bu_\varepsilon)\leqslant c_{1,\nu}+\Big(\frac{1}{p}-\frac{1}{p^*(a,b)}\Big)\big(S_{\mu,a,b}\big)^{\frac{p^*(a,b)}{p^*(a,b)-p}}-\sigma,\ \forall\,\nu\in(0,\,\nu^{**}).$$

(ii). Assume that $p\geqslant 2$ and $q>q_2$, where

$$q_2=\max\{p,\ \tfrac{N-dp^*(a,d)-(\beta(\mu)-\delta)}{\delta},\ \tfrac{N-dp^*(a,d)-(\delta-\alpha(\mu)}{\delta}\}.$$

Then

$$\begin{aligned}&N-dp^*(a,d)-q\delta<\beta(\mu)-\delta<(p-1)(\beta(\mu)-\delta),\\&N-dp^*(a,d)-q\delta<\delta-\alpha(\mu)<(p-1)(\delta-\alpha(\mu)).\end{aligned}$$

Since

$$\frac{N-dp^*(a,d)-(\beta(\mu)-\delta)}{\delta}>\frac{N-dp^*(a,d)}{\beta(\mu)},$$

from Lemma 3.1 it follows that (4.6) for some constant $\sigma>0$. Choose $\nu^{**}>0$ small enough such that $I_1<\sigma$ for $0<\nu<\nu^{**}$. Therefore (4.5) holds.

(iii). $1<p<2$, $q=p$ and $0<\lambda<\Lambda_{\mu,a,d}$. Discussing as above we also have

$$\begin{aligned}J_\nu(Au_1+Bu_\varepsilon)\leqslant c_{1,\nu}+\Big(\frac{1}{p}-\frac{1}{p^*(a,b)}\Big)\big(S_{\mu,a,b}\big)^{\frac{p^*(a,b)}{p^*(a,b)-p}}-C\int_\Omega\frac{u_\varepsilon^p}{|x|^{dp^*(a,d)}}&\\ +\ C\big(\varepsilon^{(p-1)(\delta-\alpha(\mu))}+\varepsilon^{(p-1)(\beta(\mu)-\delta)}\big)|\ln\varepsilon|+I_1.&\end{aligned}$$

If $(p-1)(\beta(\mu)-\delta)>N-dp^*(a,d)-p\delta$, then

$$\beta(\mu)>\delta+\frac{p(a+1)-dp^*(a,d)}{p-1}>\frac{N-dp^*(a,d)}{p}.$$

If $(p-1)(\delta-\alpha(\mu)) > N - dp^*(a,d) - p\delta$, then

$$\alpha(\mu) < \delta - \frac{p(a+1)-dp^*(a,d)}{p-1} < \delta.$$

On the other hand, it's easy to verify that the function

$$f(t) = (p-1)t^p - \big(N - p(a+1)\big)t^{p-1} + \mu\,, \quad t \geqslant 0,$$

has the unique minimal point δ. Furthermore, $f(t)$ is decreasing on the interval $(0,\delta)$ and is increasing on the interval $(\delta,+\infty)$.

If

$$N > \widetilde{N} := p((p+1)(a+1) - dp^*(a,d)),$$

then

$$\delta + dp^*(a,d) - p(a+1) > 0.$$

Furthermore,

$$\delta + \frac{p(a+1)-dp^*(a,d)}{p-1} < \beta(\mu),$$

which is equivalent to

$$f\big(\delta + \frac{p(a+1)-dp^*(a,d)}{p-1}\big) < f\big(\beta(\mu)\big) = 0,$$

and it is equivalent to

$$0 \leqslant \mu < \mu_1,$$

where

$$\mu_1 = \big(\delta + \tfrac{p(a+1)-dp^*(a,d)}{p-1}\big)^{p-1}(\delta + dp^*(a,d) - p(a+1)).$$

If

$$N > N' = p(a+1+\frac{p(a+1)-dp^*(a,d)}{p-1}),$$

then

$$0 < \delta - \frac{p(a+1)-dp^*(a,d)}{p-1} < \delta.$$

Hence

$$\delta - \frac{p(a+1)-dp^*(a,d)}{p-1} > \alpha(\mu),$$

which is equivalent to

$$f\big(\delta - \frac{p(a+1)-dp^*(a,d)}{p-1}\big) < f\big(\alpha(\mu)\big) = 0,$$

and it is equivalent to

$$0 \leqslant \mu < \mu_2,$$

where

$$\mu_2 = \big(\delta - \tfrac{p(a+1)-dp^*(a,d)}{p-1}\big)^{p-1}(\delta + p(a+1) - dp^*(a,d)).$$

Now we choose N and μ such that

$$N > \max\{\widetilde{N}, N'\} = N' \quad \text{and} \quad 0 \leqslant \mu < \min\{\mu_1, \mu_2\}.$$

Then taking ε small we have

$$C\big(\varepsilon^{(p-1)(\delta-\alpha(\mu))} + \varepsilon^{(p-1)(\beta(\mu)-\delta)}\big)|\ln\varepsilon| - C\int_\Omega \frac{u_\varepsilon^p}{|x|^{dp^*(a,d)}} \leqslant -2\sigma$$

for some constant $\sigma > 0$. Choose $\nu^{**} > 0$ small enough such that $I_1 < \sigma$ for $0 < \nu < \nu^{**}$. Thus (4.5) holds.

(iv). $p \geqslant 2$, $q = p$ and $0 < \lambda < \Lambda_{\mu,a,d}$. Discussing as above we also have

$$\begin{aligned} J_\nu(Au_1 + Bu_\varepsilon) \leqslant c_{1,\nu} &+ \Big(\frac{1}{p} - \frac{1}{p^*(a,b)}\Big)\big(S_{\mu,a,b}\big)^{\frac{p^*(a,b)}{p^*(a,b)-p}} \\ &+ C\big(\varepsilon^{\beta(\mu)-\delta} + \varepsilon^{\delta-\alpha(\mu)}\big)|\ln\varepsilon| - C\int_\Omega \frac{u_\varepsilon^p}{|x|^{dp^*(a,d)}} + I_1. \end{aligned}$$

If $\beta(\mu) - \delta > N - dp^*(a,d) - p\delta$, then

$$\beta(\mu) > \delta + p(a+1) - dp^*(a,d) > \frac{N - dp^*(a,d)}{p}.$$

Assume that $N > N'' = p(a+1+(p-1)(p(a+1)-dp^*(a,d)))$. Then

$$\delta + (p-1)(dp^*(a,d) - p(a+1)) > 0.$$

Furthermore,

$$\delta + p(a+1) - dp^*(a,d) < \beta(\mu),$$

which is equivalent to

$$f\big(\delta + p(a+1) - dp^*(a,d)\big) < f\big(\beta(\mu)\big) = 0,$$

and it is equivalent to

$$0 \leqslant \mu < \mu_3,$$

where

$$\mu_3 = \big(\delta + p(a+1) - dp^*(a,d)\big)^{p-1}(\delta + (p-1)(dp^*(a,d) - p(a+1))).$$

If $\delta - \alpha(\mu) > N - dp^*(a,d) - p\delta$, then

$$\alpha(\mu) < \delta - p(a+1) + dp^*(a,d) < \delta.$$

Assume $N > \bar{N} := p((p+1)(a+1) - dp^*(a,d))$. Then

$$0 < \delta - p(a+1) + dp^*(a,d) < \delta.$$

Consequently,

$$\delta - p(a+1) + dp^*(a,d) > \alpha(\mu),$$

which is equivalent to

$$f\big(\delta - p(a+1) + dp^*(a,d)\big) < f\big(\alpha(\mu)\big) = 0,$$

and it is equivalent to

$$0 \leqslant \mu < \mu_4,$$

where

$$\mu_4 = \Big(\delta - p(a+1) + dp^*(a,d)\Big)^{p-1}(\delta + (p-1)(p(a+1) - dp^*(a,d))).$$

Now we choose N and μ such that

$$N > \max\big\{ N'', \bar{N}\big\} = N'' \quad \text{and} \quad 0 \leqslant \mu < \min\{\mu_3, \mu_4\}.$$

Taking ε small we have

$$C\big(\varepsilon^{(p-1)(\delta-\alpha(\mu))} + \varepsilon^{(p-1)(\beta(\mu)-\delta)}\big)|\ln\varepsilon| - C\int_\Omega \frac{u_\varepsilon^p}{|x|^{dp^*(a,d)}} \leqslant -2\sigma$$

for some constant $\sigma > 0$. Choose $\nu^{**} > 0$ small enough such that $I_1 < \sigma$ for $0 < \nu < \nu^{**}$. Therefore (4.5) holds.

The proof of the lemma is completed.

PROOF OF THEOREM 1.1. Set $\nu_0 = \min\{\nu', \nu^*, \nu^{**}\}$. From the fact $c_{1,\nu} \to c_{1,0}$ as $\nu \to 0$ and by Lemma 4.4 it follows that $c_{2,\nu}$ is bounded uniformly in $\nu \in (0, \nu_0)$. Let $u_{2,\nu}$ be the solution obtained in Lemma 4.3. Then there exists a constant $C > 0$ such that

$$\|u_{2,\nu}\| \leqslant C, \qquad \forall \nu \in (0, \nu_0). \tag{4.7}$$

Define $u^\pm(x) = \max\{\pm u(x), 0\}$ respectively for any $u \in W_{0,a}^{1,p}(\Omega)$, then $u^\pm \in W_{0,a}^{1,p}(\Omega)$. By (4.6) we can find $\nu_n \to 0$ such that for some $u \in W_{0,a}^{1,p}(\Omega)$,

$$u_{2,\nu_n}^\pm \rightharpoonup u^\pm \text{ weakly in } W_{0,a}^{1,p}(\Omega).$$

For convenience, we denote $u_{2,\nu_n}, J_{\nu_n}, c_{1,\nu_n}, c_{2,\nu_n}$ and Λ_{ν_n} as $u_n, J_n, c_{1,n}, c_{2,n}$ and Λ_n respectively. Since $u_n^\pm \in \Lambda_n$, $J_n(u_n^\pm) \geqslant c_{1,n}$. From Lemma 4.4 it follows that

$$J_n(u_n^+) + J_n(u_n^-) = J_n(u_n) = c_{2,n} \leqslant c_{1,n} + \Big(\frac{1}{p} - \frac{1}{p^*(a,b)}\Big)\big(S_{\mu,a,b}\big)^{\frac{p^*(a,b)}{p^*(a,b)-p}} - \sigma$$

for n large. Necessarily,

$$J_n(u_n^\pm) \leqslant \Big(\frac{1}{p} - \frac{1}{p^*(a,b)}\Big)\big(S_{\mu,a,b}\big)^{\frac{p^*(a,b)}{p^*(a,b)-p}} - \sigma.$$

By (4.6) and the fact that $u_n^{\pm} \in \Lambda_n$ we derive

$$C_1 \leqslant \int_\Omega \frac{|u_n^{\pm}|^{p^*(a,b)}}{|x|^{bp^*(a,b)}} \leqslant C_2$$

for some positive constants C_1 and C_2.

Now we study the convergence of $\{u_n^{\pm}\}$. Note that $\{u_n^{\pm}\}$ is bounded in $W_{0,a}^{1,p}(\Omega)$. By Lemma 2.1 and the concentration compactness theorem ([13],[14]) and up to a subsequence, we have that $u_n^{\pm} \to u^{\pm}$ strongly in $W_{0,a}^{1,p}(\Omega)$ for some $u \in W_{0,a}^{1,p}(\Omega)$ and $u^{\pm} \not\equiv 0$. Therefore u changes sign in Ω, $u_n \rightharpoonup u$ weakly in $W_{0,a}^{1,p}(\Omega)$ and u is a solution of (1.1). Since $c_{2,n} \to c_{2,0}$ as $n \to \infty$, it is easy to verify that $\{u_n\}$ is actually a PS sequence for J_0 at level $c_{2,0}$. From $\lim_{n\to\infty} c_{1,n} = c_{1,0}$ and by standard arguments we can show that a subsequence of $\{u_n\}$ converges strongly to u in $W_{0,a}^{1,p}(\Omega)$. Therefore $c_{2,0} = J_0(u)$. The proof of Theorem 1.1 is completed. $\square$

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