

## ON THE STOKES EQUATIONS WITH THE NAVIER-TYPE BOUNDARY CONDITIONS

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*Dedicated to Professor Jesús Ildefonso Díaz  
on the occasion of his 60th birthday*

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*Abstract.* In a possibly multiply-connected three dimensional bounded domain, we prove in the  $L^p$  theory the existence and uniqueness of vector potentials, associated with a divergence-free function and satisfying non homogeneous boundary conditions. Furthermore, we consider the stationary Stokes equations with nonstandard boundary conditions of the form  $\mathbf{u} \cdot \mathbf{n} = g$  and  $\mathbf{curl} \mathbf{u} \times \mathbf{n} = \mathbf{h} \times \mathbf{n}$  on the boundary  $\Gamma$ . We prove the existence and uniqueness of weak, strong and very weak solutions. Our proofs are mainly based on *Inf – Sup* conditions.

### 1. Introduction

In this paper, we are interested in the stationary Stokes equations:

$$-\Delta \mathbf{u} + \nabla \pi = \mathbf{f} \quad \text{and} \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \quad (1.1)$$

where  $\mathbf{u}$  is the velocity vector field,  $\pi$  is the pressure,  $\mathbf{f}$  is the external force and  $\Omega \subset \mathbb{R}^3$  is bounded possibly multiply-connected domain. This system is mostly studied with no-slip Dirichlet's boundary condition, corresponding to the case where the boundary coincides with a fixed wall. However, this condition is not always realistic and gives rise to the phenomenon of strong boundary layers in general. For example, in immiscible two-phase flows, the moving contact line is not compatible with the no-slip boundary condition. Another example occurs when moderate pressure is involved such as in high altitude aerodynamics (see [21]). More generally, if the wall is smooth, the fluid can slip on the boundary. In 1827, Navier [19] proposed a slip-with-friction boundary condition, in which there is a stagnant layer of fluid close to the wall allowing a fluid to slip, and the tangential component of the strain tensor should be proportional to the tangential component of the fluid velocity on the boundary:

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{and} \quad ((\mathbb{D}\mathbf{u})\mathbf{n} + \alpha \mathbf{u})_{\boldsymbol{\tau}} = 0, \quad (1.2)$$

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where  $\mathbb{D}\mathbf{u} = (\nabla\mathbf{u} + \nabla\mathbf{u}^\top)/2$  is the strain tensor,  $\alpha$  is a scalar friction function.

Let us introduce some notations. For any vector field  $\mathbf{v}$  on  $\Gamma$ , we shall denote by  $v_n$  the component of  $\mathbf{v}$  in the direction of  $\mathbf{n}$ , while we shall denote by  $\mathbf{v}_\tau$  the projection of  $\mathbf{v}$  on the tangent hyperplane to  $\Gamma$ . In other words  $v_n = \mathbf{v} \cdot \mathbf{n}$  and  $\mathbf{v}_\tau = \mathbf{v} - v_n\mathbf{n}$ .

Let us now consider any point  $P$  on  $\Gamma$  and choose an open neighbourhood  $W$  of  $P$  in  $\Gamma$ , small enough, to allow the existence of two families of  $\mathcal{C}^2$  curves on  $W$ . The lengths  $s_1, s_2$  along each family of curves are a possible system of coordinates in  $W$ . We denote by  $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2$  the unit tangent vectors to each family of curves. With this notations, we have  $\mathbf{v}_\tau = \sum_{k=1}^2 v_k \boldsymbol{\tau}_k$ , where  $v_j = \mathbf{v} \cdot \boldsymbol{\tau}_j$ .

The conditions (1.2) are also used for simulations of flows in the presence of rough boundaries, such as in aerodynamics, or in the case of perforated boundary, which is called Beavers-Joseph’s law, or still in weather forecasts and in hemodynamics (see [6]). They are used in particular in the large eddy simulation of turbulent flows.

For a mathematical analysis of the Stokes system satisfying (1.2), the first pionnering paper is due to Solonnikov and Scadilov [22] (with  $\alpha = 0$ ). More recently, Beirao da Veiga proved existence results for weak and strong solutions in the  $L^2$  setting. However, we can prove that

$$2((\mathbb{D}\mathbf{u})\mathbf{n})_\tau = \nabla_\tau(\mathbf{u} \cdot \mathbf{n}) + \left(\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\right)_\tau - \sum_{j=1}^2 \left(\frac{\partial\mathbf{n}}{\partial s_j} \cdot \mathbf{u}_\tau\right)\boldsymbol{\tau}_j. \tag{1.3}$$

On the other hand, we have the following relation:

$$\mathbf{curl}\mathbf{u} \times \mathbf{n} = \nabla_\tau(\mathbf{u} \cdot \mathbf{n}) - \left(\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\right)_\tau - \sum_{j=1}^2 \left(\frac{\partial\mathbf{n}}{\partial s_j} \cdot \mathbf{u}_\tau\right)\boldsymbol{\tau}_j, \tag{1.4}$$

which implies that the boundary condition  $\mathbf{curl}\mathbf{u} \times \mathbf{n} = \mathbf{h}$  is equivalent to the condition  $2((\mathbb{D}\mathbf{u})\mathbf{n})_\tau + 2\sum_{j=1}^2 \left(\frac{\partial\mathbf{n}}{\partial s_j} \cdot \mathbf{u}_\tau\right)\boldsymbol{\tau}_j = -\mathbf{h}$  when  $\mathbf{u} \cdot \mathbf{n} = 0$  on  $\Gamma$ . Comparing with (1.2), the following boundary condition

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{and} \quad \mathbf{curl}\mathbf{u} \times \mathbf{n} = 0, \tag{1.5}$$

is in fact a slip-Navier boundary condition type. For more details concerning the relationship between slip-Navier boundary condition and the boundary conditions (1.5), the interested reader is referred to [7], [10] and [18].

The main propose of this paper is to develop a  $L^p$  theory for  $1 < p < \infty$  to deal with the well-posedness for the the stationary Stokes equations with the boundary conditions (1.5). The motivation of our study comes from the remark that the slip-Navier boundary condition is widely accepted in many applications and numerical studies. Another motivation is from the fact that, so far, the above problemMitrea-pipher has been attacked mainly within the framework of the Hilbert spaces (see [14]). However, in the case  $p \neq 2$ , the  $L^p(\Omega)$  theory has yet not fully developed. So, in this paper we apply many of the techniques used when  $p = 2$ , add critical new techniques for  $p \neq 2$  and provide a general framework for developing the  $L^p(\Omega)$  theory of the Stokes problem with the boundary conditions (1.5).

Stokes problem with the boundary conditions (1.5) was studied by Conca, Murat and Pironneau (*cf.* [14] and [13]) in a more general framework. They supposed that the boundary  $\Gamma$  is divided into three parts and that the boundary conditions was of three different types, where the boundary condition (1.5) was given on a portion of the boundary. They proved the existence of variational solutions  $\mathbf{u}$  and show that they were solutions of the initial Stokes problem. In particular, they suppose that Laplacian was in  $L^2(\Omega)$  without specifying any conditions on the data which would imply this regularity. Next, this studies were completed by Bernard [8], where he showed that if the pressure, which is a given data on a portion of the boundary, is more regular, the variational solution  $\mathbf{u}$  of the Stokes problem satisfies  $\Delta \mathbf{u} \in L^2(\Omega)$  and the corresponding boundary conditions. He generalized this result and he proved a regularity  $W^{m,r}(\Omega)$ ,  $m \in \mathbb{N}$ ,  $m \geq 2$ ,  $r \geq 2$  for the stationary Stokes problem and also the regularity  $W^{2,r}(\Omega)$ ,  $r \geq 2$  for the time-dependent Stokes problem in [9].

Unlike the Stokes problem with Dirichlet boundary conditions, the boundary conditions (1.5) permit the pressure to be completely decoupled to the velocity as a solution of a certain Neumann problem. Moreover, as we shall remark, the pressure can be constant under certain assumptions. So, by setting  $\mathbf{F} = \mathbf{f} - \nabla \pi$ , we obtain a system of equations involving only the velocity variable  $\mathbf{u}$ . That is

$$-\Delta \mathbf{u} = \mathbf{F} \quad \text{and} \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \quad (1.6)$$

with the boundary conditions (1.5). In [12], variational formulations in Hilbert spaces are proposed for the elliptic problem (1.6) with no divergence constraint on the spaces of test functions. This problem is also studied by [16], where their method is based on the theory of Agmon-Douglis-Nirenberg and they consider a domain with  $\mathcal{C}^\infty$ -boundary.

In Section 2, we introduce some notations and the functional spaces, besides we derive some basic Sobolev inequalities for vector fields.

In Section 3, we propose a characterization of the vector potentials related to the geometrical properties of the domain  $\Omega$ . To this effects, non homogeneous conditions on the boundary values of the vector potential on  $\Gamma$  are imposed, together with the no divergence-free condition. These results are known in the Hilbertian case, see for instance [1] for the case of homogeneous boundary conditions. This work is an extension to the non Hilbertian case and to nonhomogeneous boundary conditions in  $L^p$  theory, usually useful for solving problems with the curl curl operator. This leads to establish an Inf-Sup condition which plays a crucial role in the proof of the solvability of the elliptic problem (1.5)-(1.6). Based on this, in Section 4, we conclude with the main result of this paper related to the well-posedness of the Stokes problem (1.1)-(1.5). We prove the existence and the uniqueness of weak solutions which we shall call strong solutions under more regular assumptions on the datas. Next, we state results related to the very weak solution when irregular datas are considered. For this, we need some preliminary results including density lemmas, characterization of dual spaces and a trace's result for very weak solutions.

Note that here, we are interested only in the Stokes equations and this may be applied to the Navier-Stokes equations.

### 2. Preliminaries

Let  $\Omega$  be a bounded open connected set of  $\mathbb{R}^3$  of class  $\mathcal{C}^{1,1}$  which boundary  $\Gamma$ . Let  $\Gamma_i, 0 \leq i \leq I$ , denote the connected components of the boundary  $\Gamma$ ,  $\Gamma_0$  being the exterior boundary of  $\Omega$ . We also fix a smooth open set  $\mathcal{O}$  with a connected boundary (a ball, for instance), such that  $\bar{\Omega}$  is contained in  $\mathcal{O}$ , and we denote by  $\Omega_i, 0 \leq i \leq I$ , the connected component of  $\mathcal{O} \setminus \bar{\Omega}$  with boundary  $\Gamma_i (\Gamma_0 \cup \partial\mathcal{O}$  for  $i = 0$ ). We do not assume that  $\Omega$  is simply-connected but we suppose that there exist  $J$  connected open surfaces  $\Sigma_j, 1 \leq j \leq J$ , called 'cuts', contained in  $\Omega$ , such that each surface  $\Sigma_j$  is an open subset of a smooth manifold, the boundary of  $\Sigma_j$  is contained in  $\Gamma$ . The intersection  $\bar{\Sigma}_i \cap \bar{\Sigma}_j$  is empty for  $i \neq j$ , and finally the open set  $\Omega^\circ = \Omega \setminus \cup_{j=1}^J \Sigma_j$  is simply-connected and pseudo- $\mathcal{C}^{1,1}$  (see [5] and [20]). We denote by  $[\cdot]_j$  the jump of a function over  $\Sigma_j$ , i.e. the differences of the traces, for  $1 \leq j \leq J$  and by  $\langle \cdot, \cdot \rangle_{X, X'}$  the duality product between a space  $X$  and  $X'$ . We shall use bold characters for the vectors or the vector spaces and the non-bold characters for the scalars. The letter  $C$  denotes a constant that is not necessarily the same from an occurrences. Finally, for any function  $q$  in  $W^{1,p}(\Omega^\circ)$ ,  $\mathbf{grad} q$  is the gradient of  $q$  in the sense of distributions in  $\mathcal{D}'(\Omega^\circ)$ . It belongs to  $L^p(\Omega^\circ)$  and therefore can be extended to  $L^p(\Omega)$ . In order to distinguish this extension from the gradient of  $q$  in  $\mathcal{D}'(\Omega)$ , we denote it by  $\widetilde{\mathbf{grad}} q$ . We define the spaces:

$$\begin{aligned} \mathbf{H}^p(\mathbf{curl}, \Omega) &= \{ \mathbf{v} \in L^p(\Omega); \mathbf{curl} \mathbf{v} \in L^p(\Omega) \}, \\ \mathbf{H}^p(\text{div}, \Omega) &= \{ \mathbf{v} \in L^p(\Omega); \text{div} \mathbf{v} \in L^p(\Omega) \}, \\ \mathbf{X}^p(\Omega) &= \mathbf{H}^p(\mathbf{curl}, \Omega) \cap \mathbf{H}^p(\text{div}, \Omega), \end{aligned}$$

equipped with the graph norm. now, we also define their subspaces:

$$\begin{aligned} \mathbf{H}_0^p(\mathbf{curl}, \Omega) &= \{ \mathbf{v} \in \mathbf{H}^p(\mathbf{curl}, \Omega); \mathbf{v} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma \}, \\ \mathbf{H}_0^p(\text{div}, \Omega) &= \{ \mathbf{v} \in \mathbf{H}^p(\text{div}, \Omega); \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma \}, \\ \mathbf{X}_N^p(\Omega) &= \{ \mathbf{v} \in \mathbf{X}^p(\Omega); \mathbf{v} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma \}, \\ \mathbf{X}_T^p(\Omega) &= \{ \mathbf{v} \in \mathbf{X}^p(\Omega); \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma \}. \end{aligned}$$

Let us state some results which are used throughout this paper. Let us have a look at the case  $\Omega$  simply connected ( $\Sigma_j = \emptyset, 1 \leq j \leq J$ ). Problem (1.1) with (1.5) has an equivalent variational formulation for any  $p$ . Indeed, if  $\mathbf{u} \in \mathbf{V}_T^p(\Omega)$  is solution of (1.1) with (1.5), then  $\mathbf{u} \in \mathbf{W}^{1,p}(\Omega)$  satisfies:

$$\forall \mathbf{v} \in \mathbf{V}_T^{p'}(\Omega), \int_{\Omega} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \mathbf{v} \, dx = \langle \mathbf{f}, \mathbf{v} \rangle_{\Omega}, \tag{2.1}$$

where

$$\mathbf{V}_T^p(\Omega) = \{ \mathbf{w} \in \mathbf{X}_T^p(\Omega); \text{div} \mathbf{w} = 0 \text{ in } \Omega \text{ and } \langle \mathbf{w} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, 1 \leq j \leq J \}.$$

Observe that, due to the boundary conditions, we can not take  $\mathbf{f} \in \mathbf{W}^{-1,p}(\Omega)$ . Then, we must choose an adequate distribution space for  $\mathbf{f}$  which is given here by the dual

space  $(\mathbf{H}_0^{p'}(\operatorname{div}, \Omega))'$  and this amounts to defining the duality pairing in (2.1). Observe that  $(\mathbf{H}_0^{p'}(\operatorname{div}, \Omega))' \hookrightarrow \mathbf{W}^{-1,p}(\Omega)$  and for any  $\mathbf{f}$ , there exist  $\boldsymbol{\psi} \in \mathbf{L}^p(\Omega)$  and  $\chi \in L^p(\Omega)$  such that (see [2])

$$\mathbf{f} = \boldsymbol{\psi} + \nabla \chi.$$

First, recall the case when  $p = 2$ , for any  $\mathbf{v} \in \mathbf{V}_T^2(\Omega)$ , we have:

$$\|\mathbf{v}\|_{\mathbf{H}^1(\Omega)} \leq C \|\operatorname{curl} \mathbf{v}\|_{\mathbf{L}^2(\Omega)}. \tag{2.2}$$

This equivalence of the semi-norm  $\|\operatorname{curl} \cdot\|_{\mathbf{L}^2(\Omega)}$  and the full norm  $\|\cdot\|_{\mathbf{H}^1(\Omega)}$  and the Lax-Milgram Lemma allow to prove that problem (2.1) has a unique solution  $\mathbf{u} \in \mathbf{H}^1(\Omega)$ . We point out here that the inequality (2.2) is quite simple for  $p = 2$  while for  $p \neq 2$  and  $\Omega$  multiply connected, it is not obvious and the situation is different. It is known that for any vector field with vanishing trace on the boundary, we have for any  $1 < p < \infty$  and for a bounded open set  $\Omega$  of  $\mathbb{R}^3$  with boundary  $\Gamma$  of class  $\mathcal{C}^{1,1}$  the following inequality:

$$\|\nabla \mathbf{v}\|_{\mathbf{L}^p(\Omega)} \leq C(\|\operatorname{div} \mathbf{v}\|_{L^p(\Omega)} + \|\operatorname{curl} \mathbf{v}\|_{\mathbf{L}^p(\Omega)}). \tag{2.3}$$

However, in the case of vectors fields with either vanishing tangential components or vanishing normal components on the boundary, the inequality (2.3) is not true. So, we are interested in some inequalities of type (2.3), when  $\Omega$  has arbitrary Betti numbers and for vectors fields with vanishing tangential components or vanishing normal components on the boundary. This is because

$$\begin{aligned} \mathbf{K}_N^p(\Omega) &= \{\mathbf{v} \in \mathbf{L}^p(\Omega), \operatorname{div} \mathbf{v} = 0, \operatorname{curl} \mathbf{v} = \mathbf{0} \text{ in } \Omega \text{ and } \mathbf{v} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma\}, \\ \mathbf{K}_T^p(\Omega) &= \{\mathbf{v} \in \mathbf{L}^p(\Omega), \operatorname{div} \mathbf{v} = 0, \operatorname{curl} \mathbf{v} = \mathbf{0} \text{ in } \Omega \text{ and } \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma\} \end{aligned}$$

have dimensions the first Betti number  $I \geq 1$  and second Betti number  $J \geq 1$  respectively. As shown in [1, Proposition 3.14], when  $p = 2$ , we can prove for any  $1 < p < \infty$  that the space  $\mathbf{K}_T^p(\Omega)$  is spanned by the functions  $\widetilde{\operatorname{grad}} q_j^T$ ,  $1 \leq j \leq J$ , where each  $q_j^T \in W^{1,p}(\Omega^\circ)$  is unique up to an additive constant and satisfies:

$$\begin{cases} \Delta q_j^T = 0 \text{ in } \Omega^\circ, \quad \partial_n q_j^T = 0 \text{ on } \Gamma, \quad [q_j^T]_k = \text{constant}, \\ [\partial_n q_j^T]_k = 0; \quad 1 \leq k \leq J \text{ and } \langle \partial_n q_j^T, 1 \rangle_{\Sigma_k} = \delta_{jk}, \quad 1 \leq k \leq J. \end{cases}$$

We note that  $\mathbf{K}_T^p(\Omega) = \{0\}$  if  $J = 0$ . For more details see [5] and [20]. Similarly, we can prove that the space  $\mathbf{K}_N^p(\Omega)$  is  $I$  and that it is spanned by the functions  $\operatorname{grad} q_i^N$ ,  $1 \leq i \leq I$ , where each  $q_i^N \in W^{1,p}(\Omega)$  is the unique solution to the problem:

$$\begin{cases} \Delta q_i^N = 0 \text{ in } \Omega, \quad q_i^N = 0 \text{ in } \Gamma_0, \quad q_i^N = \text{constant in } \Gamma_k, \\ \langle \partial_n q_i^N, 1 \rangle_{\Gamma_0} = -1 \text{ and } \langle \partial_n q_i^N, 1 \rangle_{\Gamma_k} = \delta_{ik}, \quad 1 \leq k \leq I. \end{cases}$$

We note that if  $\Gamma = \Gamma_0$ , then  $\mathbf{K}_N^p(\Omega) = \{0\}$ .

By means of the integral representation formula for  $\mathbf{v} \in \mathbf{W}^{1,p}(\Omega)$  and by introducing the both integral operators defined by:

$$T \lambda(\mathbf{x}) = -\frac{1}{2\pi} \int_{\Gamma} \lambda(\boldsymbol{\xi}) \frac{\partial}{\partial \mathbf{n}} |\mathbf{x} - \boldsymbol{\xi}|^{-1} d\sigma_{\boldsymbol{\xi}},$$

$$R \lambda(\mathbf{x}) = \frac{1}{2\pi} \int_{\Gamma} \mathbf{curl} \left( \frac{\lambda(\boldsymbol{\xi})}{|\mathbf{x} - \boldsymbol{\xi}|} \right) \times \mathbf{n} d\sigma_{\boldsymbol{\xi}},$$

we can prove, for every  $1 < p < \infty$ , the following first inequality concerning tangential vector fields:

$$\|\nabla \mathbf{v}\|_{L^p(\Omega)} \leq C(\|\operatorname{div} \mathbf{v}\|_{L^p(\Omega)} + \|\mathbf{curl} \mathbf{v}\|_{L^p(\Omega)} + \sum_{j=1}^J |\langle \mathbf{v} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j}|), \tag{2.4}$$

and the second one concerns the normal vector fields:

$$\|\nabla \mathbf{v}\|_{L^p(\Omega)} \leq C(\|\operatorname{div} \mathbf{v}\|_{L^p(\Omega)} + \|\mathbf{curl} \mathbf{v}\|_{L^p(\Omega)} + \sum_{i=1}^I |\langle \mathbf{v} \cdot \mathbf{n}, 1 \rangle_{\Gamma_i}|). \tag{2.5}$$

Von Wahl [23] obtained (2.4) and (2.5) without any flux through the cuts  $\Sigma_j$  ( $1 \leq j \leq J$ ) and the components  $\Gamma_i$  ( $1 \leq i \leq I$ ) on the right hand sides. So, he proved that such homogeneous estimates hold if and only if  $I = 0$ , *i.e.*  $\Omega$  is simply connected in the case of  $\mathbf{v} \times \mathbf{n} = \mathbf{0}$ , and if and only if  $J = 0$ , *i.e.*  $\Omega$  has only one connected component of the boundary  $\Gamma$  in the case  $\mathbf{u} \cdot \mathbf{n} = 0$  on  $\Gamma$ , respectively. In [11], the authors prove  $C^\alpha$ -estimates of type (2.4) and (2.5) in a bounded smooth open set.

Using (2.4) and (2.5), the density of  $\mathbf{W}^{1,p}(\Omega) \cap \mathbf{X}_N^p(\Omega)$  in  $\mathbf{X}_N^p(\Omega)$  and the density of  $\mathbf{W}^{1,p}(\Omega) \cap \mathbf{X}_T^p(\Omega)$  in  $\mathbf{X}_T^p(\Omega)$ , we obtain the following continuous embeddings:

$$\mathbf{X}_N^p(\Omega) \hookrightarrow \mathbf{W}^{1,p}(\Omega) \quad \text{and} \quad \mathbf{X}_T^p(\Omega) \hookrightarrow \mathbf{W}^{1,p}(\Omega).$$

In order to consider the case of nonhomogeneous boundary conditions, we introduce the following spaces:

$$\mathbf{X}^{1,p}(\Omega) = \{\mathbf{v} \in L^p(\Omega); \operatorname{div} \mathbf{v} \in L^p(\Omega), \mathbf{curl} \mathbf{v} \in L^p(\Omega), \mathbf{v} \cdot \mathbf{n} \in \mathbf{W}^{1-\frac{1}{p},p}(\Gamma)\}$$

and

$$\mathbf{Y}^{1,p}(\Omega) = \{\mathbf{v} \in L^p(\Omega); \operatorname{div} \mathbf{v} \in W^p(\Omega), \mathbf{curl} \mathbf{v} \in L^p(\Omega), \mathbf{v} \times \mathbf{n} \in \mathbf{W}^{1-\frac{1}{p},p}(\Omega)\}.$$

**THEOREM 2.1.** *The spaces  $\mathbf{X}^{1,p}(\Omega)$  and  $\mathbf{Y}^{1,p}(\Omega)$  are both continuously imbedded in  $\mathbf{W}^{1,p}(\Omega)$ :*

i) any  $\mathbf{v}$  in  $\mathbf{X}^{1,p}(\Omega)$  satisfies

$$\|\mathbf{v}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C(\|\mathbf{v}\|_{L^p(\Omega)} + \|\mathbf{curl} \mathbf{v}\|_{L^p(\Omega)} + \|\operatorname{div} \mathbf{v}\|_{L^p(\Omega)} + \|\mathbf{v} \cdot \mathbf{n}\|_{\mathbf{W}^{1-\frac{1}{p},p}(\Gamma)});$$

ii) any  $\mathbf{v}$  in  $\mathbf{Y}^{1,p}(\Omega)$  satisfies

$$\|\mathbf{v}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C(\|\mathbf{v}\|_{L^p(\Omega)} + \|\mathbf{curl} \mathbf{v}\|_{L^p(\Omega)} + \|\operatorname{div} \mathbf{v}\|_{L^p(\Omega)} + \|\mathbf{v} \times \mathbf{n}\|_{\mathbf{W}^{1-\frac{1}{p},p}(\Gamma)}).$$

Now, due to the Peetre-Tartar Theorem, we deduce the following first Poincaré’s inequality for every function  $\mathbf{u} \in \mathbf{W}^{1,p}(\Omega)$  with  $\mathbf{u} \times \mathbf{n} = \mathbf{0}$  on  $\Gamma$ :

$$\|\mathbf{w}\|_{L^p(\Omega)} \leq C(\|\mathbf{curl}\mathbf{w}\|_{L^p(\Omega)} + \|\mathbf{div}\mathbf{w}\|_{L^p(\Omega)} + \sum_{i=1}^I |\langle \mathbf{w} \cdot \mathbf{n}, 1 \rangle_{\Gamma_i}|)$$

and a second one for every function  $\mathbf{u} \in \mathbf{W}^{1,p}(\Omega)$  with  $\mathbf{u} \cdot \mathbf{n} = 0$  on  $\Gamma$ :

$$\|\mathbf{w}\|_{L^p(\Omega)} \leq C(\|\mathbf{curl}\mathbf{w}\|_{L^p(\Omega)} + \|\mathbf{div}\mathbf{w}\|_{L^p(\Omega)} + \sum_{j=1}^J |\langle \mathbf{w} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j}|).$$

Moreover, we shall show the corresponding estimates for  $\mathbf{v}$  in higher order Sobolev spaces  $\mathbf{W}^{m,p}(\Omega)$  with  $m \in \mathbb{N}^*$  via  $\mathbf{div}\mathbf{u}$  and  $\mathbf{curl}\mathbf{u}$  when  $\mathbf{v} \times \mathbf{n}$  or  $\mathbf{v} \cdot \mathbf{n}$  does not vanish on  $\Gamma$ . For example, when  $m = 2$ , we prove the following inequalities (see [5] and [20]):

$$\|\mathbf{v}\|_{\mathbf{W}^{2,p}(\Omega)} \leq C(\|\mathbf{v}\|_{L^p(\Omega)} + \|\mathbf{curl}\mathbf{v}\|_{\mathbf{W}^{1,p}(\Omega)} + \|\mathbf{div}\mathbf{v}\|_{\mathbf{W}^{1,p}(\Omega)} + \|\mathbf{v} \cdot \mathbf{n}\|_{\mathbf{W}^{1-\frac{1}{p},p}(\Gamma)}),$$

$$\|\mathbf{v}\|_{\mathbf{W}^{2,p}(\Omega)} \leq C(\|\mathbf{v}\|_{L^p(\Omega)} + \|\mathbf{curl}\mathbf{v}\|_{\mathbf{W}^{1,p}(\Omega)} + \|\mathbf{div}\mathbf{v}\|_{\mathbf{W}^{1,p}(\Omega)} + \|\mathbf{v} \times \mathbf{n}\|_{\mathbf{W}^{1-\frac{1}{p},p}(\Gamma)}),$$

where we use the properties of derivatives on the boundary  $\Gamma$ . These inequalities will be useful in order to prove regularity results of solution of the Stokes problem and elliptic problems that we will solve.

### 3. Vector potentials with non homogeneous boundary conditions

This section relies on some results concerning vector potentials with non homogeneous boundary conditions. So we prove that a divergence-free vector field is the curl of a divergence-free vector field called vector potential, satisfying two kinds of non homogeneous boundary conditions. We also prove the existence of other types of vector potentials which no longer divergence-free. These results are of great importance for the study of the Stokes equation with different boundary conditions. First, we state the following result which is going to be useful for us in the sequel.

LEMMA 3.1. *A vector field  $\mathbf{u}$  in  $\mathbf{H}^p(\mathbf{div}, \Omega)$  satisfies*

$$\mathbf{div}\mathbf{u} = 0 \quad \text{in } \Omega \quad \text{and} \quad \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Gamma_i} = 0, \quad 0 \leq i \leq I, \tag{3.1}$$

*if and only if there exists a vector potential  $\boldsymbol{\psi}_0$  in  $\mathbf{W}^{1,p}(\Omega)$  such that*

$$\mathbf{u} = \mathbf{curl}\boldsymbol{\psi}_0. \tag{3.2}$$

*Moreover, we can choose  $\boldsymbol{\psi}_0$  such that  $\mathbf{div}\boldsymbol{\psi}_0 = 0$  and we have the estimate*

$$\|\boldsymbol{\psi}_0\|_{\mathbf{W}^{1,p}(\Omega)} \leq C\|\mathbf{u}\|_{L^p(\Omega)}, \tag{3.3}$$

*where  $C > 0$  depends only on  $p$  and  $\Omega$ .*

*Proof.* **1.** The necessity of conditions (3.1) can be established exactly with the same arguments as in [1].

**2.** Conversely, let  $\mathbf{u}$  be any function satisfying (3.1). The idea is to extend  $\mathbf{u}$  to the whole space so that the extended function  $\tilde{\mathbf{u}}$  belongs to  $L^p(\mathbb{R}^3)$ , is divergence-free and has a compact support. Then, it will be easy to construct its stream function by means of the fundamental solution of the Laplacian. Let then  $\chi_0$  in  $W^{1,p}(\Omega)$  be the unique solution up to an additive constant of the following Neumann problem

$$-\Delta\chi_0 = 0 \text{ in } \Omega_0 \text{ and } \partial_n\chi_0 = \mathbf{u} \cdot \mathbf{n} \text{ on } \Gamma_0, \quad \partial_n\chi_0 = 0 \text{ on } \partial\mathcal{O},$$

(see the introduction for the notations), and let  $\chi_i \in W^{1,p}(\Omega)$  with  $1 \leq i \leq I$ , be the unique solution up to an additive constant of the problem:

$$-\Delta\chi_i = 0 \text{ in } \Omega_i \text{ and } \partial_n\chi_i = \mathbf{u} \cdot \mathbf{n} \text{ on } \Gamma_i,$$

with the estimate:

$$\|\chi_i\|_{W^{1,p}(\Omega_i)} \leq C\|\mathbf{u}\|_{L^p(\Omega)},$$

and where  $\mathbf{n}$  denotes the unit outward normal to  $\Omega$  and  $\mathcal{O}$ . Now we can extend  $\mathbf{u}$  as follows

$$\tilde{\mathbf{u}} = \begin{cases} \mathbf{u} & \text{in } \Omega, \\ \mathbf{grad} \chi_i & \text{in } \Omega_i, \quad 0 \leq i \leq I, \\ \mathbf{0} & \text{in } \mathbb{R}^3 \setminus \bar{\mathcal{O}}. \end{cases}$$

Clearly,  $\tilde{\mathbf{u}}$  belongs to  $H^p(\text{div}, \mathbb{R}^3)$  and is divergence-free in  $\mathbb{R}^3$ . The function  $\boldsymbol{\psi}_0 = \mathbf{curl}(E * \tilde{\mathbf{u}})$ , with  $E$  the fundamental solution of the laplacian, satisfies

$$\mathbf{curl} \boldsymbol{\psi}_0 = \tilde{\mathbf{u}} \text{ and } \text{div} \boldsymbol{\psi}_0 = 0 \text{ in } \mathbb{R}^3.$$

Applying the Calderón Zygmund inequality, we obtain

$$\|\nabla \boldsymbol{\psi}_0\|_{L^p(\mathbb{R}^3)} \leq C\|\Delta(E * \tilde{\mathbf{u}})\|_{L^p(\mathbb{R}^3)} \leq C\|\tilde{\mathbf{u}}\|_{L^p(\mathbb{R}^3)} \leq C\|\mathbf{u}\|_{L^p(\Omega)}.$$

Due to [3, Proposition 2.10],  $\boldsymbol{\psi}_0|_\Omega$  belongs to  $W^{1,p}(\Omega)$ . As a consequence,  $\boldsymbol{\psi}_0$  satisfies the condition (3.2) and the estimate (3.3).  $\square$

**REMARK 3.2.** A detailed proof of the case  $p = 2$  can be found in [1, Lemma 3.5] and [15, Theorem 3.4] by using the Fourier transformation.

Our first theorem deals with nonhomogeneous normal boundary condition.

**THEOREM 3.3.** *Let  $g \in W^{-1/p,p}(\Gamma)$  and  $\chi \in L^p(\Omega)$  satisfies*

$$\int_\Omega \chi \, \mathbf{dx} = \langle g, 1 \rangle_\Gamma. \tag{3.4}$$

*A function  $\mathbf{u} \in L^p(\Omega)$  satisfies*

$$\text{div} \mathbf{u} = 0 \text{ in } \Omega \text{ and } \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Gamma_i} = 0, \quad 1 \leq i \leq I, \tag{3.5}$$

if and only if there exists a vector potential  $\boldsymbol{\psi}$  in  $L^p(\Omega)$  such that

$$\begin{aligned} \mathbf{u} &= \mathbf{curl} \boldsymbol{\psi} \quad \text{and} \quad \operatorname{div} \boldsymbol{\psi} = \chi \text{ in } \Omega, \\ \boldsymbol{\psi} \cdot \mathbf{n} &= g \quad \text{on } \Gamma, \\ \langle \boldsymbol{\psi} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} &= 0 \text{ for any } 1 \leq j \leq J. \end{aligned} \tag{3.6}$$

This function  $\boldsymbol{\psi}$  is unique and we have the estimate:

$$\|\boldsymbol{\psi}\|_{L^p(\Omega)} \leq C(\|\mathbf{u}\|_{L^p(\Omega)} + \|g\|_{W^{-1/p,p}(\Gamma)} + \|\chi\|_{L^p(\Omega)}). \tag{3.7}$$

If moreover  $g \in W^{1-1/p,p}(\Gamma)$ , then  $\boldsymbol{\psi} \in \mathbf{W}^{1,p}(\Omega)$  with the corresponding estimate.

*Proof.* The uniqueness and the necessity of conditions (3.5) can be established exactly as in [1]. Next, with a function  $\mathbf{u} \in L^p(\Omega)$  satisfying (3.5), we first construct the vector potential  $\boldsymbol{\psi}_1$  satisfying:

$$\begin{aligned} \mathbf{u} &= \mathbf{curl} \boldsymbol{\psi}_1 \quad \text{and} \quad \operatorname{div} \boldsymbol{\psi}_1 = 0 \text{ in } \Omega, \\ \boldsymbol{\psi}_1 \cdot \mathbf{n} &= 0 \quad \text{on } \Gamma, \quad \langle \boldsymbol{\psi}_1 \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, \quad 1 \leq j \leq J. \end{aligned} \tag{3.8}$$

According to Lemma 3.1, there exists  $\boldsymbol{\psi}_0 \in \mathbf{W}^{1,p}(\Omega)$  with  $\operatorname{div} \boldsymbol{\psi}_0 = 0$  in  $\Omega$ , such that  $\mathbf{u} = \mathbf{curl} \boldsymbol{\psi}_0$  and with

$$\|\boldsymbol{\psi}_0\|_{\mathbf{W}^{1,p}(\Omega)} \leq C\|\mathbf{u}\|_{L^p(\Omega)}.$$

We introduce the solution  $\chi$  in  $W^{2,p}(\Omega)$ , unique up to an additive constant, of the problem:

$$-\Delta \chi = 0 \text{ in } \Omega, \quad \text{and} \quad \partial_n \chi = \boldsymbol{\psi}_0 \cdot \mathbf{n} \text{ on } \Gamma,$$

which satisfies the estimate

$$\|\chi\|_{W^{2,p}(\Omega)/\mathbb{R}} \leq C\|\boldsymbol{\psi}_0 \cdot \mathbf{n}\|_{W^{1-1/p,p}(\Gamma)}.$$

Then, we set  $\boldsymbol{\psi}_1 = \boldsymbol{\psi}_0 + \nabla(\theta - \chi)$ , with  $\theta \in W^{1,p}(\Omega)$  solution, up to an additive constant, of the Neumann problem:

$$\Delta \theta = \chi \text{ in } \Omega \quad \text{and} \quad \frac{\partial \theta}{\partial \mathbf{n}} = g \text{ on } \Gamma,$$

with the estimate:

$$\|\theta\|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C(\|\chi\|_{L^p(\Omega)} + \|g\|_{W^{-1/p,p}(\Gamma)}). \tag{3.9}$$

Finally, the function

$$\widetilde{\boldsymbol{\psi}} = \boldsymbol{\psi} - \sum_{j=1}^J \langle \boldsymbol{\psi} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} \widetilde{\mathbf{grad}} q_j^T.$$

belongs to  $L^p(\Omega)$  and satisfies (3.6).

We suppose now that  $g \in W^{1-1/p,p}(\Gamma)$ . Applying Theorem 2.1, we have immediately  $\widetilde{\boldsymbol{\psi}} \in \mathbf{W}^{1,p}(\Omega)$ .  $\square$

REMARK 3.4. If  $\Omega$  is Lipschitz and simply connected, the first part of the previous theorem was proved by Mitrea [17] if  $3/2 - \varepsilon \leq p \leq 2 + \varepsilon$  for some  $\varepsilon$  strictly positive.

As a consequence, we can prove the following Inf-Sup condition.

COROLLARY 3.5. *The following Inf-Sup condition holds: there exists a constant  $\beta > 0$ , such that*

$$\inf_{\substack{\boldsymbol{\varphi} \in \mathbf{V}_T^{p'}(\Omega) \\ \boldsymbol{\varphi} \neq 0}} \sup_{\substack{\boldsymbol{\xi} \in \mathbf{V}_T^p(\Omega) \\ \boldsymbol{\xi} \neq 0}} \frac{\int_{\Omega} \mathbf{curl} \boldsymbol{\xi} \cdot \mathbf{curl} \boldsymbol{\varphi} \, dx}{\|\boldsymbol{\xi}\|_{X_T^p(\Omega)} \|\boldsymbol{\varphi}\|_{X_T^{p'}(\Omega)}} \geq \beta. \tag{3.10}$$

*Proof.* The proof is based on the decomposition of  $L^p(\Omega)$  into the direct sum of solenoidal vector fields and gradients of scalar functions together with the result of Lemma 3.1.  $\square$

Next, we give the corresponding result of vector potentials in the case of non homogeneous normal boundary condition. As previously, we first establish the existence of a divergence free vector potential.

THEOREM 3.6. *Let  $\mathbf{u} \in L^p(\Omega)$  and  $\mathbf{g}$  such that  $\mathbf{g} \times \mathbf{n} \in \mathbf{W}^{-1/p,p}(\Gamma)$ . Then*

$$\begin{cases} \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} \cdot \mathbf{n} = \operatorname{div}_{\Gamma}(\mathbf{g} \times \mathbf{n}) & \text{on } \Gamma, \\ \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, & 1 \leq j \leq J, \end{cases} \tag{3.11}$$

if and only if there exists a vector potential  $\boldsymbol{\psi}$  in  $L^p(\Omega)$  such that

$$\begin{aligned} \mathbf{u} &= \mathbf{curl} \boldsymbol{\psi} \quad \text{and} \quad \operatorname{div} \boldsymbol{\psi} = 0 \quad \text{in } \Omega, \\ \boldsymbol{\psi} \times \mathbf{n} &= \mathbf{g} \times \mathbf{n} \quad \text{on } \Gamma, \\ \langle \boldsymbol{\psi} \cdot \mathbf{n}, 1 \rangle_{\Gamma_i} &= 0 \quad \text{for any } 1 \leq i \leq I. \end{aligned} \tag{3.12}$$

This function  $\boldsymbol{\psi}$  is unique and we have the estimate:

$$\|\boldsymbol{\psi}\|_{L^p(\Omega)} \leq C(\|\mathbf{u}\|_{L^p(\Omega)} + \|\mathbf{g} \times \mathbf{n}\|_{\mathbf{W}^{-1/p,p}(\Gamma)}). \tag{3.13}$$

If moreover  $\mathbf{g} \times \mathbf{n} \in \mathbf{W}^{1-1/p,p}(\Gamma)$ , then  $\boldsymbol{\psi} \in \mathbf{W}^{1,p}(\Omega)$  with the corresponding estimate.

*Proof.* THE FIRST STEP: NECESSARY CONDITIONS. Let us show that (3.12) implies (3.11). Clearly,  $\operatorname{div}(\mathbf{curl} \boldsymbol{\psi}) = 0$ . Next, we must check that  $\mathbf{u} \cdot \mathbf{n} = \operatorname{div}_{\Gamma}(\mathbf{g} \times \mathbf{n})$  on  $\Gamma$  and that  $\langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0$ , for any  $1 \leq j \leq J$ . Since  $\boldsymbol{\psi} \in L^p(\Omega)$  and  $\mathbf{curl} \boldsymbol{\psi} \in L^p(\Omega)$ , we have for any  $\chi$  in  $W^{2,p'}(\Omega)$ :

$$\begin{aligned} \int_{\Omega} \mathbf{curl} \boldsymbol{\psi} \cdot \mathbf{grad} \chi \, dx &= \langle \mathbf{u} \cdot \mathbf{n}, \chi \rangle_{\mathbf{W}^{-1/p,p}(\Gamma) \times \mathbf{W}^{1-1/p',p'}(\Gamma)}, \\ \int_{\Omega} \mathbf{curl} \boldsymbol{\psi} \cdot \mathbf{grad} \chi \, dx &= -\langle \boldsymbol{\psi} \times \mathbf{n}, \mathbf{grad} \chi \rangle_{\mathbf{W}^{-1/p,p}(\Gamma) \times \mathbf{W}^{1-1/p',p'}(\Gamma)} \end{aligned}$$

$$= \langle \operatorname{div}_\Gamma(\boldsymbol{\psi} \times \mathbf{n}), \chi \rangle_{W^{-1-1/p,p}(\Gamma) \times W^{2-1/p',p'}(\Gamma)}.$$

As for any function  $\chi$  in  $W^{2,p'}(\Omega)$ ,

$$\langle \mathbf{u} \cdot \mathbf{n}, \chi \rangle_\Gamma = \langle \operatorname{div}_\Gamma(\mathbf{g} \times \mathbf{n}), \chi \rangle_\Gamma,$$

we deduce the trace's result. The necessity of the last condition in (3.11) can be established exactly with the same arguments as in [1].

**THE SECOND STEP: UNIQUENESS.** The proof is simple consequence of the characterization of the kernel  $\mathbf{K}_N^p(\Omega)$ .

**THE THIRD STEP: EXISTENCE.** Let  $\mathbf{u} \in \mathbf{L}^p(\Omega)$  be any function satisfying (3.11). According to Lemma 3.1, there exists  $\boldsymbol{\psi}_0 \in \mathbf{W}^{1,p}(\Omega)$  such that

$$\mathbf{u} = \operatorname{curl} \boldsymbol{\psi}_0 \quad \text{and} \quad \operatorname{div} \boldsymbol{\psi}_0 = 0 \quad \text{in } \Omega.$$

Using Lemma 3.5, the following problem: find  $\boldsymbol{\xi} \in \mathbf{V}_T^p(\Omega)$  such that for any  $\boldsymbol{\varphi} \in \mathbf{V}_T^{p'}(\Omega)$

$$\int_\Omega \operatorname{curl} \boldsymbol{\xi} \cdot \operatorname{curl} \boldsymbol{\varphi} \, dx = \int_\Omega \boldsymbol{\psi}_0 \cdot \operatorname{curl} \boldsymbol{\varphi} \, dx - \int_\Omega \operatorname{curl} \boldsymbol{\psi}_0 \cdot \boldsymbol{\varphi} \, dx - \langle \mathbf{g} \times \mathbf{n}, \boldsymbol{\varphi} \rangle_\Gamma, \quad (3.14)$$

has a unique solution.

Note that the right-hand side defines an element of  $(\mathbf{V}_T^p(\Omega))'$ . We want to extend (3.14) to any test function  $\tilde{\boldsymbol{\varphi}}$  in  $\mathbf{X}_T^{p'}(\Omega)$ . Let  $\chi$  in  $W^{2,p'}(\Omega)$  satisfying:

$$\Delta \chi = \operatorname{div} \tilde{\boldsymbol{\varphi}} \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial \chi}{\partial \mathbf{n}} = 0 \quad \text{on } \Gamma, \quad (3.15)$$

and  $\boldsymbol{\varphi} \in \mathbf{V}_T^{p'}(\Omega)$  satisfying:

$$\boldsymbol{\varphi} = \tilde{\boldsymbol{\varphi}} - \operatorname{grad} \chi - \sum_{j=1}^J \langle (\tilde{\boldsymbol{\varphi}} - \operatorname{grad} \chi) \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} \widetilde{\operatorname{grad}} q_j^T. \quad (3.16)$$

Observe that

$$\int_\Omega \operatorname{curl} \boldsymbol{\psi}_0 \cdot \operatorname{grad} \chi \, dx = \int_\Omega \mathbf{u} \cdot \operatorname{grad} \chi \, dx = \langle \operatorname{div}_\Gamma(\mathbf{g} \times \mathbf{n}), \chi \rangle_\Gamma = -\langle \mathbf{g} \times \mathbf{n}, \nabla \chi \rangle_\Gamma,$$

and we obtain

$$\begin{aligned} \int_\Omega \operatorname{curl} \boldsymbol{\psi}_0 \cdot \widetilde{\operatorname{grad}} q_j^T \, dx &= \int_{\Omega^\circ} \mathbf{u} \cdot \operatorname{grad} q_j^T \, dx \\ &= \sum_{k=1}^J [q_j^T]_k \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_k} + \langle \mathbf{u} \cdot \mathbf{n}, q_j^T \rangle_\Gamma \end{aligned}$$

$$= \langle \mathbf{g} \times \mathbf{n}, \nabla q_j^T \rangle_\Gamma.$$

Hence, (3.14) becomes: find  $\boldsymbol{\xi} \in \mathbf{V}_T^p(\Omega)$  such that for any  $\tilde{\boldsymbol{\varphi}} \in \mathbf{X}_T^{p'}(\Omega)$ :

$$\int_\Omega \mathbf{curl} \boldsymbol{\xi} \cdot \mathbf{curl} \tilde{\boldsymbol{\varphi}} \, dx = \int_\Omega \boldsymbol{\psi}_0 \cdot \mathbf{curl} \tilde{\boldsymbol{\varphi}} \, dx - \int_\Omega \mathbf{curl} \boldsymbol{\psi}_0 \cdot \tilde{\boldsymbol{\varphi}} \, dx - \langle \mathbf{g} \times \mathbf{n}, \tilde{\boldsymbol{\varphi}} \rangle_\Gamma. \quad (3.17)$$

Then, every solution of (3.17) also solves the problem

$$\begin{cases} -\Delta \boldsymbol{\xi} = \mathbf{0}, & \operatorname{div} \boldsymbol{\xi} = 0 & \text{in } \Omega, \\ \boldsymbol{\xi} \cdot \mathbf{n} = 0, & (\boldsymbol{\psi}_0 - \mathbf{curl} \boldsymbol{\xi}) \times \mathbf{n} = \mathbf{g} \times \mathbf{n} & \text{on } \Gamma, \\ \langle \boldsymbol{\xi} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, & 1 \leq j \leq J. \end{cases}$$

Finally, we set

$$\boldsymbol{\psi} = \boldsymbol{\psi}_0 - \mathbf{curl} \boldsymbol{\xi} - \sum_{i=1}^I \langle (\boldsymbol{\psi}_0 - \mathbf{curl} \boldsymbol{\xi}) \cdot \mathbf{n}, 1 \rangle_{\Gamma_i} \mathbf{grad} q_i^N,$$

and it follows that the function  $\boldsymbol{\psi}$  belongs to  $L^p(\Omega)$  and satisfies (3.12) and the estimate (3.13). Observe that  $\boldsymbol{\xi} \in \mathbf{W}^{2,p}(\Omega)$  when  $\mathbf{g} \times \mathbf{n} \in \mathbf{W}^{1-1/p,p}(\Gamma)$  and then  $\boldsymbol{\psi} \in \mathbf{W}^{1,p}(\Omega)$ .  $\square$

Here also, we can extend the previous result for the case where the divergence of the vector potentials does not vanish.

**COROLLARY 3.7.** *Let  $\mathbf{g} \times \mathbf{n} \in \mathbf{W}^{-1/p,p}(\Gamma)$  and  $\chi \in W^{-1,p}(\Omega)$ . A function  $\mathbf{u} \in L^p(\Omega)$  satisfies (3.11) if and only if there exists a vector potential  $\boldsymbol{\psi}$  in  $L^p(\Omega)$  such that*

$$\begin{aligned} \mathbf{u} &= \mathbf{curl} \boldsymbol{\psi} \quad \text{and} \quad \operatorname{div} \boldsymbol{\psi} = \chi \quad \text{in } \Omega, \\ \boldsymbol{\psi} \times \mathbf{n} &= \mathbf{g} \times \mathbf{n} \quad \text{on } \Gamma, \\ \langle \boldsymbol{\psi} \cdot \mathbf{n}, 1 \rangle_{\Gamma_i} &= 0 \quad \text{for any } 1 \leq i \leq I. \end{aligned} \quad (3.18)$$

This function  $\boldsymbol{\psi}$  is unique and we have the estimate:

$$\|\boldsymbol{\psi}\|_{L^p(\Omega)} \leq C(\|\mathbf{u}\|_{L^p(\Omega)} + \|\mathbf{g} \times \mathbf{n}\|_{\mathbf{W}^{-1/p,p}(\Gamma)} + \|\chi\|_{W^{-1,p}(\Omega)}). \quad (3.19)$$

If moreover  $\mathbf{g} \times \mathbf{n} \in \mathbf{W}^{1-1/p,p}(\Gamma)$  and  $\chi \in L^p(\Omega)$ , then  $\boldsymbol{\psi} \in \mathbf{W}^{1,p}(\Omega)$  with the corresponding estimate.

*Proof. i)* Assume that (3.11) holds and let  $\boldsymbol{\psi}_0$  denote the function associated with  $\mathbf{u}$  by Theorem 3.6. We introduce the unique solution  $\theta \in W^{1,p}(\Omega)$  of the problem:

$$\Delta \theta = \chi \quad \text{in } \Omega \quad \text{and} \quad \theta = 0 \quad \text{on } \Gamma$$

satisfying the estimate:

$$\|\theta\|_{W^{1,p}(\Omega)} \leq C\|\chi\|_{W^{-1,p}(\Omega)}. \quad (3.20)$$

Finally, we set  $\boldsymbol{\psi} = \boldsymbol{\psi}_0 + \nabla \theta$ . It is easy to check that  $\boldsymbol{\psi}$  satisfies (3.18) without the last condition. Finally, the function

$$\tilde{\boldsymbol{\psi}} = \boldsymbol{\psi} - \sum_{i=1}^I \langle \boldsymbol{\psi} \cdot \mathbf{n}, 1 \rangle_{\Gamma_i} \mathbf{grad} q_i^N$$

is the required function and this concludes the proof.

ii) The uniqueness is a consequence of the characterization of the kernel  $\mathbf{K}_N^p(\Omega)$ .

iii) The necessity of conditions (3.11) was established in the proof of Theorem 3.6.  $\square$

REMARK 3.8. If  $\Omega$  is Lipschitz and simply connected, the first part of the previous theorem was proved by Mitrea [17] for  $\mathbf{g} \times \mathbf{n} \in \mathbf{L}^p(\Gamma)$  if  $3/2 - \varepsilon \leq p \leq 2 + \varepsilon$ , for some  $\varepsilon$  strictly positive.

### 4. Well-posedness of the Stokes equations

In this section we will study the following Stokes equations:

$$(\mathcal{S}_T) \begin{cases} -\Delta \mathbf{u} + \nabla \pi = \mathbf{f} & \text{and } \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} \cdot \mathbf{n} = g & \text{and } \mathbf{curl} \mathbf{u} \times \mathbf{n} = \mathbf{h} \times \mathbf{n} & \text{on } \Gamma, \\ \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, & 1 \leq j \leq J. \end{cases}$$

#### 4.1. Weak solutions

The aim of this subsection is to give a variational formulation of problem  $(\mathcal{S}_T)$  and prove a theorem of existence and uniqueness of weak solutions.

Let us consider the following space

$$\mathbf{E}^p(\Omega) = \{ \mathbf{v} \in \mathbf{W}^{1,p}(\Omega), \Delta \mathbf{v} \in [\mathbf{H}_0^{p'}(\operatorname{div}, \Omega)]' \},$$

which is a Banach space for the norm

$$\| \mathbf{v} \|_{\mathbf{E}^p(\Omega)} = \| \mathbf{v} \|_{\mathbf{W}^{1,p}(\Omega)} + \| \Delta \mathbf{v} \|_{[\mathbf{H}_0^{p'}(\operatorname{div}, \Omega)]'}$$

We have the following preliminary results. We skip the proof because it is classical.

LEMMA 4.1.  $\mathcal{D}(\overline{\Omega})$  is dense in  $\mathbf{E}^p(\Omega)$ .

As a consequence, we have the following result.

COROLLARY 4.2. The linear mapping  $\gamma : \mathbf{v} \rightarrow \mathbf{curl} \mathbf{v}|_{\Gamma} \times \mathbf{n}$  defined on  $\mathcal{D}(\overline{\Omega})$  can be extended to a linear continuous mapping

$$\gamma : \mathbf{E}^p(\Omega) \longrightarrow \mathbf{W}^{-\frac{1}{p},p}(\Gamma).$$

Moreover, we have the Green formula: for any  $\mathbf{v} \in E^p(\Omega)$  and  $\boldsymbol{\varphi} \in X_T^{p'}(\Omega)$  with  $\text{div } \boldsymbol{\varphi} = 0$  in  $\Omega$ ,

$$-\langle \Delta \mathbf{v}, \boldsymbol{\varphi} \rangle_{[\mathbf{H}_0^{p'}(\text{div}, \Omega)]' \times \mathbf{H}_0^{p'}(\text{div}, \Omega)} = \int_{\Omega} \mathbf{curl} \mathbf{v} \cdot \mathbf{curl} \boldsymbol{\varphi} \, dx - \langle \mathbf{curl} \mathbf{v} \times \mathbf{n}, \boldsymbol{\varphi} \rangle_{\Gamma}, \quad (4.1)$$

where the duality on  $\Gamma$  is defined by  $\langle \cdot, \cdot \rangle_{\Gamma} = \langle \cdot, \cdot \rangle_{\mathbf{W}^{-\frac{1}{p}, p}(\Gamma) \times \mathbf{W}^{\frac{1}{p}, p'}(\Gamma)}$ .

PROPOSITION 4.3. Let  $\mathbf{f}$  belongs to  $L^p(\Omega)$  with  $\text{div } \mathbf{f} = 0$  in  $\Omega$ ,  $g \in W^{1-\frac{1}{p}, p}(\Gamma)$  and  $\mathbf{h} \times \mathbf{n} \in \mathbf{W}^{-\frac{1}{p}, p}(\Gamma)$  verifying the following compatibility conditions: for any  $\mathbf{v} \in \mathbf{K}_T^{p'}(\Omega)$ ,

$$\int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx + \langle \mathbf{h} \times \mathbf{n}, \mathbf{v} \rangle_{\mathbf{W}^{-\frac{1}{p}, p}(\Gamma) \times \mathbf{W}^{\frac{1}{p}, p'}(\Gamma)} = 0, \quad (4.2)$$

$$\int_{\Gamma} g \, d\boldsymbol{\sigma} = 0, \quad (4.3)$$

$$\mathbf{f} \cdot \mathbf{n} - \text{div}_{\Gamma}(\mathbf{h} \times \mathbf{n}) = 0 \text{ on } \Gamma, \quad (4.4)$$

where  $\text{div}_{\Gamma}$  is the surface divergence on  $\Gamma$ . Then, the problem

$$(E_T) \quad \begin{cases} -\Delta \boldsymbol{\xi} = \mathbf{f} & \text{and } \text{div } \boldsymbol{\xi} = 0 & \text{in } \Omega, \\ \boldsymbol{\xi} \cdot \mathbf{n} = g & \text{and } \mathbf{curl} \boldsymbol{\xi} \times \mathbf{n} = \mathbf{h} \times \mathbf{n} & \text{on } \Gamma, \\ \langle \boldsymbol{\xi} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, & 1 \leq j \leq J, \end{cases}$$

has a unique solution  $\boldsymbol{\xi}$  in  $\mathbf{W}^{1,p}(\Omega)$  satisfying the estimate:

$$\|\boldsymbol{\xi}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C \left\{ \|\mathbf{f}\|_{L^p(\Omega)} + \|g\|_{W^{1-1/p,p}(\Gamma)} + \|\mathbf{h} \times \mathbf{n}\|_{\mathbf{W}^{-1/p,p}(\Gamma)} \right\}. \quad (4.5)$$

*Proof.* We check separately the existence and the uniqueness and we prove the necessary conditions (4.2)-(4.4) to establish the existence of a solution of  $(E_T)$ .

STEP 1: UNIQUENESS. Let  $\boldsymbol{\xi} \in \mathbf{W}^{1,p}(\Omega)$  be a solution of problem  $(E_T)$  with data  $\mathbf{f} = \mathbf{0}$ ,  $g = 0$  and  $\mathbf{h} = \mathbf{0}$ . The function  $\mathbf{w} = \mathbf{curl} \boldsymbol{\xi}$  belongs to  $L^p(\Omega)$  and satisfies:

$$\text{div } \mathbf{w} = 0, \quad \mathbf{curl} \mathbf{w} = \mathbf{0} \text{ in } \Omega \quad \text{and} \quad \mathbf{w} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma.$$

This implies that  $\mathbf{w} \in \mathbf{K}_N^p(\Omega)$ . Thanks to the characterization of the kernel  $\mathbf{K}_N^p(\Omega)$ , we can write:

$$\mathbf{w} = \sum_{i=1}^I \langle \mathbf{w} \cdot \mathbf{n}, 1 \rangle_{\Gamma_i} \mathbf{grad} q_i^N.$$

So,  $\mathbf{w} \in L^2(\Omega)$  and

$$\int_{\Omega} |\mathbf{w}|^2 \, dx = \sum_{i=1}^I \langle \mathbf{w} \cdot \mathbf{n}, 1 \rangle_{\Gamma_i} \int_{\Omega} \mathbf{curl} \boldsymbol{\xi} \cdot \mathbf{grad} q_i^N \, dx = 0.$$

Hence,  $\mathbf{w}$  is equal to zero and  $\boldsymbol{\xi}$  belongs to  $\mathbf{K}_T^p(\Omega)$ . Using the characterization of the kernel  $\mathbf{K}_T^p(\Omega)$  and the last condition in  $(E_T)$  yields that  $\boldsymbol{\xi}$  is also equal to zero. This yields the uniqueness of the solution of the problem  $(E_T)$ .

STEP 2: COMPATIBILITY CONDITIONS. Firstly, let us show that the conditions (4.3) and (4.4) are necessary. First, we set  $\mathbf{z} = \mathbf{curl} \boldsymbol{\xi}$ , with  $\boldsymbol{\xi}$  a solution of  $(E_T)$ . It is clear that

$$\forall \varphi \in W^{2,p'}(\Omega), \quad \langle \mathbf{curl} \mathbf{z} \cdot \mathbf{n}, \varphi \rangle_\Gamma = -\langle \mathbf{z} \times \mathbf{n}, \nabla \varphi \rangle_\Gamma,$$

where the bracket denote the duality  $W^{-1/p,p}(\Gamma) \times W^{1/p,p'}(\Gamma)$ . So that  $\mathbf{f}$  must satisfy:

$$\langle \mathbf{f} \cdot \mathbf{n}, \varphi \rangle_\Gamma = -\langle \mathbf{h} \times \mathbf{n}, \nabla \varphi \rangle_\Gamma = \langle \text{div}_\Gamma(\mathbf{h} \times \mathbf{n}), \varphi \rangle_\Gamma.$$

This shows that

$$\mathbf{f} \cdot \mathbf{n} - \text{div}_\Gamma(\mathbf{h} \times \mathbf{n}) = 0 \quad \text{in the sens of } W^{-1-\frac{1}{p},p}(\Gamma).$$

We deduce in fact that  $\text{div}_\Gamma(\mathbf{h} \times \mathbf{n})$  belongs to  $W^{-\frac{1}{p},p}(\Gamma)$  and the above equation occurs to the sens of the last space. Next, the fact that  $\int_\Gamma g \, d\sigma = 0$  is due to  $\text{div} \boldsymbol{\xi} = 0$  in  $\Omega$ . Secondly, let us show that the compatibility condition (4.2) is necessary. In a preliminary step, we lift the boundary condition on  $\boldsymbol{\xi} \cdot \mathbf{n}$  by solving the Neumann problem:

$$(\mathcal{N}) \quad \Delta \theta = 0 \text{ in } \Omega \quad \text{and} \quad \frac{\partial \theta}{\partial \mathbf{n}} = g \text{ on } \Gamma.$$

Owing to (4.3), this problem has a solution  $\theta \in W^{2,p}(\Omega)$ , unique up to an additive constant, satisfying the estimate:

$$\|\theta\|_{W^{2,p}(\Omega)/\mathbb{R}} \leq C \|g\|_{W^{1-1/p,p}(\Gamma)}. \tag{4.6}$$

The function  $\mathbf{z} = \boldsymbol{\xi} - \nabla \theta$  satisfies:

$$\begin{cases} -\Delta \mathbf{z} = \mathbf{f} & \text{and} & \text{div} \mathbf{z} = 0 & \text{in } \Omega, \\ \mathbf{z} \cdot \mathbf{n} = 0 & \text{and} & \mathbf{curl} \mathbf{z} \times \mathbf{n} = \mathbf{h} \times \mathbf{n} & \text{on } \Gamma. \end{cases} \tag{4.7}$$

Immediately, by using (4.1) yields that  $\forall \varphi \in \mathbf{V}_T^{p'}(\Omega)$ ,

$$\int_\Omega \mathbf{curl} \mathbf{z} \cdot \mathbf{curl} \varphi \, dx = \int_\Omega \mathbf{f} \cdot \varphi \, dx + \langle \mathbf{h} \times \mathbf{n}, \varphi \rangle_{W^{-\frac{1}{p},p}(\Gamma) \times W^{\frac{1}{p},p'}(\Gamma)}, \tag{4.8}$$

and we deduce the compatibility condition (4.2).

STEP 3: EXISTENCE. If we prove that the problem (4.7) has a unique solution  $\mathbf{z} \in \mathbf{W}^{1,p}(\Omega)$ , then  $\boldsymbol{\xi} = \mathbf{z} + \nabla \theta \in \mathbf{W}^{1,p}(\Omega)$  is the required solution of  $(E_T)$ . As a first step, we check that the problem (4.8) is well-posed, without referring to the initial problem (4.7). We know from Lemma 3.5, that the problem (4.8) satisfies the Inf-Sup condition (3.10). So, it has a unique solution  $\mathbf{z} \in \mathbf{V}_T^p(\Omega) \subset \mathbf{W}^{1,p}(\Omega)$  since the right-hand sides defines an element of  $(\mathbf{V}_T^{p'}(\Omega))'$ . Next, we want to extend (4.8) to any

test function  $\tilde{\boldsymbol{\varphi}}$  in  $\mathbf{X}_T^{p'}(\Omega)$ . We consider the solution  $\chi$  in  $W^{1,p'}(\Omega)$ , unique up to an additive constant, of the Neumann problem:

$$\Delta \chi = \operatorname{div} \tilde{\boldsymbol{\varphi}} \text{ in } \Omega \quad \text{and} \quad \frac{\partial \chi}{\partial n} = 0 \text{ on } \Gamma, \tag{4.9}$$

and we set

$$\boldsymbol{\varphi} = \tilde{\boldsymbol{\varphi}} - \operatorname{grad} \chi - \sum_{j=1}^J \langle (\tilde{\boldsymbol{\varphi}} - \operatorname{grad} \chi) \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} \widetilde{\operatorname{grad} q_j^T}. \tag{4.10}$$

Observe that  $\boldsymbol{\varphi}$  belongs to  $\mathbf{V}_T^{p'}(\Omega)$  and  $\operatorname{curl} \boldsymbol{\varphi} = \operatorname{curl} \tilde{\boldsymbol{\varphi}}$ . Moreover, using the compatibility conditions (4.2) and (4.4), we obtain:

$$\forall \chi \in W^{1,p'}(\Omega), \quad \int_{\Omega} \mathbf{f} \cdot \nabla \chi \, dx + \langle \mathbf{h} \times \mathbf{n}, \nabla \chi \rangle_{\Gamma} = \langle \mathbf{f} \cdot \mathbf{n} - \operatorname{div}_{\Gamma}(\mathbf{h} \times \mathbf{n}), \chi \rangle_{\Gamma} = 0,$$

and for  $1 \leq j \leq J$ ,

$$\int_{\Omega} \mathbf{f} \cdot \widetilde{\operatorname{grad} q_j^T} \, dx + \langle \mathbf{h} \times \mathbf{n}, \widetilde{\operatorname{grad} q_j^T} \rangle_{\Gamma} = 0.$$

So, there exists a unique solution  $\boldsymbol{\xi} \in \mathbf{V}_T^p(\Omega)$  such that

$$\forall \tilde{\boldsymbol{\varphi}} \in \mathbf{X}_T^{p'}(\Omega), \quad \int_{\Omega} \operatorname{curl} \boldsymbol{\xi} \cdot \operatorname{curl} \tilde{\boldsymbol{\varphi}} \, dx = \int_{\Omega} \mathbf{f} \cdot \tilde{\boldsymbol{\varphi}} \, dx + \langle \mathbf{h} \times \mathbf{n}, \tilde{\boldsymbol{\varphi}} \rangle_{\Gamma}. \tag{4.11}$$

Then,

$$\operatorname{curl} \operatorname{curl} \boldsymbol{\xi} = \mathbf{f} \text{ in } \Omega.$$

Since  $\boldsymbol{\xi}$  belongs to the space  $\mathbf{V}_T^p(\Omega)$  we have  $\operatorname{div} \boldsymbol{\xi} = 0$  in  $\Omega$ ,  $\boldsymbol{\xi} \cdot \mathbf{n} = 0$  on  $\Gamma$  and  $\langle \boldsymbol{\xi} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0$ ,  $1 \leq j \leq J$ . Then, it remains to verify the boundary condition

$$\operatorname{curl} \boldsymbol{\xi} \times \mathbf{n} = \mathbf{h} \times \mathbf{n} \text{ on } \Gamma.$$

We multiply the equation  $-\Delta \boldsymbol{\xi} = \mathbf{f}$  in  $\Omega$  by  $\boldsymbol{\varphi} \in \mathbf{X}_T^{p'}(\Omega)$ , we integrate on  $\Omega$  and we compare with (4.11). Consequently, for any  $\boldsymbol{\varphi} \in \mathbf{X}_T^{p'}(\Omega)$  we obtain:

$$\langle \operatorname{curl} \boldsymbol{\xi} \times \mathbf{n}, \boldsymbol{\varphi} \rangle_{\Gamma} = \langle \mathbf{h} \times \mathbf{n}, \boldsymbol{\varphi} \rangle_{\Gamma}.$$

Let now  $\boldsymbol{\mu}$  any element of the space  $\mathbf{W}^{1-\frac{1}{p'},p'}(\Gamma)$ . So, there exists an element  $\boldsymbol{\varphi}$  of  $\mathbf{W}^{1,p'}(\Omega)$  such that  $\boldsymbol{\varphi} = \boldsymbol{\mu}_t$  on  $\Gamma$ , where  $\boldsymbol{\mu}_t$  is the tangential component of  $\boldsymbol{\mu}$  on  $\Gamma$ . It is clear that  $\boldsymbol{\varphi}$  belongs to  $\mathbf{X}_T^{p'}(\Omega)$  and

$$\begin{aligned} \langle \operatorname{curl} \boldsymbol{\xi} \times \mathbf{n}, \boldsymbol{\mu} \rangle_{\Gamma} - \langle \mathbf{h} \times \mathbf{n}, \boldsymbol{\mu} \rangle_{\Gamma} &= \langle \operatorname{curl} \boldsymbol{\xi} \times \mathbf{n}, \boldsymbol{\mu}_t \rangle_{\Gamma} - \langle \mathbf{h} \times \mathbf{n}, \boldsymbol{\mu}_t \rangle_{\Gamma} \\ &= \langle \operatorname{curl} \boldsymbol{\xi} \times \mathbf{n}, \boldsymbol{\varphi} \rangle_{\Gamma} - \langle \mathbf{h} \times \mathbf{n}, \boldsymbol{\varphi} \rangle_{\Gamma} \\ &= 0. \end{aligned}$$

This implies that  $\operatorname{curl} \boldsymbol{\xi} \times \mathbf{n} = \mathbf{h} \times \mathbf{n}$  on  $\Gamma$  and then  $\boldsymbol{\xi}$  is a solution of problem (4.7) which satisfies the estimate (4.5).  $\square$

We now can solve the Stokes problem  $(\mathcal{S}_T)$ .

THEOREM 4.4. (Weak solutions for  $(\mathcal{S}_T)$ ) *Let*

$$\mathbf{f} \in [\mathbf{H}_0^{p'}(\operatorname{div}, \Omega)]', \quad g \in W^{1-\frac{1}{p}, p}(\Gamma), \quad \mathbf{h} \times \mathbf{n} \in W^{-\frac{1}{p}, p}(\Gamma) \tag{4.12}$$

and verifying the compatibility conditions (4.2)-(4.3). Then, the Stokes problem  $(\mathcal{S}_T)$  has a unique solution  $(\mathbf{u}, \pi) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$  satisfying the estimate:

$$\|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)/\mathbb{R}} \leq (\|\mathbf{f}\|_{[\mathbf{H}_0^{p'}(\operatorname{div}, \Omega)]'} + \|g\|_{W^{1-\frac{1}{p}, p}(\Gamma)} + \|\mathbf{h} \times \mathbf{n}\|_{W^{-\frac{1}{p}, p}(\Gamma)}). \tag{4.13}$$

*Proof.* STEP 1: UNIQUENESS. Let  $(\mathbf{u}, \pi)$  be a solution of  $(\mathcal{S}_T)$  with data  $\mathbf{f} = \mathbf{0}$ ,  $g = 0$  and  $\mathbf{h} = \mathbf{0}$ . Since  $\Delta \mathbf{u} \cdot \mathbf{n} = 0$  on  $\Gamma$ , we find that  $\pi$  satisfies:

$$\Delta \pi = 0 \text{ in } \Omega \quad \text{and} \quad \frac{\partial \pi}{\partial \mathbf{n}} = \mathbf{0} \text{ on } \Gamma.$$

We deduce that  $\pi = \text{constant}$  in  $\Omega$  and this implies the uniqueness up to an additive constant of the pressure. So,  $\mathbf{u}$  is a solution of a problem of type of  $(E_T)$  still with data  $\mathbf{f} = \mathbf{0}$ ,  $g = 0$  and  $\mathbf{h} = \mathbf{0}$ . Using the same uniqueness argument in Proposition 4.3, we check that  $\mathbf{u}$  is equal to zero and this yields the uniqueness of the solution of problem  $(\mathcal{S}_T)$ .

STEP 2: COMPATIBILITY CONDITIONS. We note that condition (4.3) is necessary, because  $\operatorname{div} \mathbf{u} = 0$  in  $\Omega$ . Now, let us justify the necessary of condition (4.2). Before, using the solution  $\theta \in W^{2,p}(\Omega)$  of the Neumann problem  $(\mathcal{N})$ , we lift the boundary condition on  $\mathbf{u} \cdot \mathbf{n}$ . Thus, the function  $\mathbf{z} = \mathbf{u} - \nabla \theta$  satisfies:

$$\begin{cases} -\Delta \mathbf{z} + \nabla \pi = \mathbf{f} & \text{and} & \operatorname{div} \mathbf{z} = 0 & \text{in} \\ \Omega, & & & \\ \mathbf{z} \cdot \mathbf{n} = 0 & \text{and} & \operatorname{curl} \mathbf{z} \times \mathbf{n} = \mathbf{h} \times \mathbf{n} & \text{on } \Gamma. \end{cases} \tag{4.14}$$

It is easy to prove that the function  $\mathbf{z}$  is also solution of (4.8) where we replace the integral  $\int_{\Omega} \mathbf{f} \cdot \boldsymbol{\varphi} \, dx$  by the brackets

$$\langle \cdot, \cdot \rangle_{[\mathbf{H}_0^{p'}(\operatorname{div}, \Omega)]' \times \mathbf{H}_0^{p'}(\operatorname{div}, \Omega)}$$

As a consequence, the compatibility condition (4.2) comes from (4.8) by taking  $\boldsymbol{\varphi} \in \mathbf{K}_T^{p'}(\Omega)$ .

STEP 3: EXISTENCE. The proof is similar to that of Proposition 4.3. We know that problem (4.8) has a unique solution  $\mathbf{z} \in \mathbf{V}_T^p(\Omega)$  satisfying the estimate:

$$\|\mathbf{z}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C\{\|\mathbf{f}\|_{[\mathbf{H}_0^{p'}(\operatorname{div}, \Omega)]'} + \|\mathbf{h} \times \mathbf{n}\|_{W^{-1/p, p}(\Gamma)}\}. \tag{4.15}$$

Next, we want to extend (4.8) to any test function in  $\mathbf{X}_T^{p'}(\Omega)$  with  $\operatorname{div} \tilde{\boldsymbol{\varphi}} = 0$  in  $\Omega$ . For any function  $\tilde{\boldsymbol{\varphi}} \in \mathbf{X}_T^{p'}(\Omega)$  with  $\operatorname{div} \tilde{\boldsymbol{\varphi}} = 0$  in  $\Omega$ , we set

$$\boldsymbol{\varphi} = \widetilde{\boldsymbol{\varphi}} - \sum_{j=1}^J \langle \boldsymbol{\varphi} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} \widetilde{\mathbf{grad}} q_j^T.$$

It is clear that  $\boldsymbol{\varphi}$  belongs to  $V_T^{p'}(\Omega)$ . Using the compatibility condition (4.2), for any  $\widetilde{\boldsymbol{\varphi}} \in X_T^{p'}(\Omega)$  with  $\text{div } \widetilde{\boldsymbol{\varphi}} = 0$  in  $\Omega$ , we have:

$$\int_{\Omega} \mathbf{curlz} \cdot \mathbf{curl} \widetilde{\boldsymbol{\varphi}} \, dx = \langle \mathbf{f}, \widetilde{\boldsymbol{\varphi}} \rangle_{(\mathbf{H}_0^{p'}(\text{div}, \Omega))' \times \mathbf{H}_0^{p'}(\text{div}, \Omega)} - \langle \mathbf{h} \times \mathbf{n}, \widetilde{\boldsymbol{\varphi}} \rangle_{W^{-\frac{1}{p}, p}(\Gamma) \times W^{\frac{1}{p}, p'}(\Gamma)}. \tag{4.16}$$

Taking  $\widetilde{\boldsymbol{\varphi}} \in \mathcal{D}_{\sigma}(\Omega)$  as the test function in (4.16), we obtain:

$$\langle -\Delta \mathbf{z} - \mathbf{f}, \widetilde{\boldsymbol{\varphi}} \rangle_{(\mathcal{D}(\Omega))' \times \mathcal{D}(\Omega)} = 0.$$

By De Rham theorem, there exists a function  $\pi \in L^p(\Omega)$  such that

$$-\Delta \mathbf{z} + \nabla \pi = \mathbf{f} \quad \text{in } \Omega.$$

Moreover, by the fact that  $\mathbf{z}$  belongs to the space  $V_T^p(\Omega)$  we have  $\text{div } \mathbf{z} = 0$  in  $\Omega$ ,  $\mathbf{z} \cdot \mathbf{n} = 0$  on  $\Gamma$  and  $\langle \mathbf{z} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, 1 \leq j \leq J$ . The remainder boundary condition

$$\mathbf{curlz} \times \mathbf{n} = \mathbf{h} \times \mathbf{n} \quad \text{on } \Gamma$$

is implicitly contained in (4.16) and can be derived by observing that since  $\mathbf{f}$  and  $\nabla \pi$  are two elements of  $[\mathbf{H}_0^{p'}(\text{div}, \Omega)]'$ , it is the same for  $\Delta \mathbf{z}$ . Otherwise, the space  $\mathcal{D}_{\sigma}(\Omega)$  of divergence free functions of  $\mathcal{D}(\Omega)$  is dense in the subspace of  $\mathbf{H}_0^{p'}(\text{div}, \Omega)$  with divergence free, it is clear then that for any  $\boldsymbol{\varphi} \in \mathbf{H}_0^{p'}(\text{div}, \Omega)$  with  $\text{div } \boldsymbol{\varphi} = 0$ :

$$\langle \nabla \pi, \boldsymbol{\varphi} \rangle_{[\mathbf{H}_0^{p'}(\text{div}, \Omega)]' \times \mathbf{H}_0^{p'}(\text{div}, \Omega)} = 0.$$

As a consequence, the pair  $(\mathbf{z}, \pi) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$  is the unique solution of the problem (4.14). Finally  $(\mathbf{u} = \mathbf{z} + \nabla \theta, \pi) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$  is the unique solution of the problem  $(\mathcal{S}_T)$ , where  $\theta \in W^{2,p}(\Omega)$  is solution of the problem  $(\mathcal{N})$ . The estimate (4.13) is easily derived from the construction of  $(\mathbf{u}, \pi)$ .  $\square$

REMARK 4.5. Observe that if we suppose in Theorem 4.4 that  $\mathbf{f} \in L^p(\Omega)$  with  $\text{div } \mathbf{f} = 0$  in  $\Omega$  and we add the compatibility condition (4.4), then the pressure  $\pi$  is constant. Indeed, from the first equation in the Stokes problem  $(\mathcal{S}_T)$ , we obtain that

$$\Delta \pi = 0 \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial \pi}{\partial \mathbf{n}} = \mathbf{f} \cdot \mathbf{n} - \text{div}_{\Gamma}(\mathbf{h} \times \mathbf{n}) = 0 \quad \text{on } \Gamma.$$

This implies that  $\pi$  is a constant and the Stokes problem  $(\mathcal{S}_T)$  is nothing other than problem  $(E_T)$ .

REMARK 4.6. We can also solve the Stokes problem when the divergence operator does not vanish and it is a given function. With the same assumptions Theorem 4.4, with  $\chi \in L^p(\Omega)$ , the problem:

$$\begin{cases} -\Delta \mathbf{u} + \nabla \pi = \mathbf{f} \text{ and } \operatorname{div} \mathbf{u} = \chi & \text{in } \Omega, \\ \mathbf{u} \cdot \mathbf{n} = g \text{ and } \operatorname{curl} \mathbf{u} \times \mathbf{n} = \mathbf{h} \times \mathbf{n} & \text{on } \Gamma, \\ \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, \quad 1 \leq j \leq J, \end{cases} \quad (4.17)$$

has a unique solution  $(\mathbf{u}, \pi) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$ .

### 4.2. Strong solutions

We prove now existence of strong solutions  $(\mathbf{u}, \pi) \in \mathbf{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$  for the Stokes problem  $(\mathcal{S}_T)$  and this will be used in the duality argument needed to show existence of very weak solutions. The first result that we need is the existence of strong solutions for  $(E_T)$ .

PROPOSITION 4.7. *Let*

$$\mathbf{f} \in L^p(\Omega), \quad g \in W^{2-\frac{1}{p},p}(\Gamma), \quad \mathbf{h} \times \mathbf{n} \in \mathbf{W}^{1-\frac{1}{p},p}(\Gamma) \quad (4.18)$$

with  $\operatorname{div} \mathbf{f} = 0$  in  $\Omega$  and satisfying the compatibility conditions (4.2)-(4.4). Then, the solution  $\boldsymbol{\xi}$  of the problem  $(E_T)$  given by Proposition 4.3 belongs to  $\mathbf{W}^{2,p}(\Omega)$  and satisfies the estimate:

$$\|\boldsymbol{\xi}\|_{\mathbf{W}^{2,p}(\Omega)} \leq (\|\mathbf{f}\|_{L^p(\Omega)} + \|g\|_{W^{2-\frac{1}{p},p}(\Gamma)} + \|\mathbf{h} \times \mathbf{n}\|_{\mathbf{W}^{1-\frac{1}{p},p}(\Gamma)}). \quad (4.19)$$

*Proof.* Let  $\boldsymbol{\xi} \in \mathbf{W}^{1,p}(\Omega)$  be the solution of problem  $(E_T)$  given by Proposition 4.3 and we set  $\mathbf{z} = \operatorname{curl} \boldsymbol{\xi}$ . Then,  $\mathbf{z}$  satisfies:

$$\mathbf{z} \in L^p(\Omega), \quad \operatorname{curl} \mathbf{z} \in L^p(\Omega), \quad \operatorname{div} \mathbf{z} \in L^p(\Omega) \quad \text{and} \quad \mathbf{z} \times \mathbf{n} \in \mathbf{W}^{1-\frac{1}{p},p}(\Gamma).$$

It follows from Theorem 2.1 that  $\mathbf{z}$  belongs to  $\mathbf{W}^{1,p}(\Omega)$ . Since  $g \in W^{2-\frac{1}{p},p}(\Gamma)$ , we deduce directly by using again Theorem 2.1 that  $\boldsymbol{\xi} \in \mathbf{W}^{2,p}(\Omega)$ .  $\square$

In analogy with Proposition 4.7, we derive the following regularity result for the solution of the problem  $(\mathcal{S}_T)$ .

THEOREM 4.8. (Strong solutions for  $(\mathcal{S}_T)$ ) *Let  $\mathbf{f}, g, \mathbf{h}$  satisfying (4.18) and the compatibility conditions (4.2)-(4.3). Then, the solution  $(\mathbf{u}, \pi)$  of problem  $(\mathcal{S}_T)$  given by Theorem 4.4 belongs to  $\mathbf{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$  and satisfies the estimate:*

$$\|\mathbf{u}\|_{\mathbf{W}^{2,p}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C (\|\mathbf{f}\|_{L^p(\Omega)} + \|g\|_{W^{2-\frac{1}{p},p}(\Gamma)} + \|\mathbf{h} \times \mathbf{n}\|_{\mathbf{W}^{1-\frac{1}{p},p}(\Gamma)}). \quad (4.20)$$

*Proof.* We note that under the hypothesis of Theorem 4.8, the data  $\mathbf{f}$ ,  $g$  and  $\mathbf{h}$  also satisfy the hypothesis of Theorem 4.4. So, this implies that problem  $(\mathcal{S}_T)$  has a unique solution  $(\mathbf{u}, \pi) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$ . Moreover, since  $\pi$  satisfies:

$$\operatorname{div}(\nabla \pi - \mathbf{f}) = 0 \quad \text{in } \Omega, \quad (\nabla \pi - \mathbf{f}) \cdot \mathbf{n} = -\operatorname{div}_\Gamma(\mathbf{h} \times \mathbf{n}) \quad \text{on } \Gamma, \quad (4.21)$$

then,  $\pi \in W^{1,p}(\Omega)$ . Now, we set  $\mathbf{z} = \operatorname{curl} \mathbf{u}$ . Then,  $\mathbf{z}$  satisfies:

$$\mathbf{z} \in L^p(\Omega), \operatorname{div} \mathbf{z} = 0 \text{ in } \Omega, \quad \operatorname{curl} \mathbf{z} = \nabla \pi - \mathbf{f} \in L^p(\Omega), \quad \mathbf{z} \times \mathbf{n} = \mathbf{h} \times \mathbf{n} \text{ on } \Gamma.$$

Thanks to Theorem 2.1, the function  $\mathbf{z}$  belongs to  $\mathbf{W}^{1,p}(\Omega)$ . As a consequence, due to Theorem 2.1, the solution  $\mathbf{u}$  of the problem  $(\mathcal{S}_T)$  belongs to  $\mathbf{W}^{2,p}(\Omega)$ .  $\square$

**COROLLARY 4.9.** *Let  $\mathbf{f} \in [\mathbf{H}_0^{p'}(\operatorname{div}, \Omega)]'$ ,  $g = 0$  and  $\mathbf{h} = \mathbf{0}$  on  $\Gamma$  with  $\mathbf{f}$  satisfying the compatibility condition*

$$\forall \boldsymbol{\varphi} \in \mathbf{K}_N^{p'}(\Omega), \quad \langle \mathbf{f}, \boldsymbol{\varphi} \rangle_{[\mathbf{H}_0^{p'}(\operatorname{div}, \Omega)]' \times \mathbf{H}_0^{p'}(\operatorname{div}, \Omega)} = 0.$$

*Then, the solution  $(\mathbf{u}, \pi)$  of problem  $(\mathcal{S}_T)$  given by Theorem 4.4 belongs to  $\mathbf{W}^{2,p}(\Omega) \times L^p(\Omega)$ .*

*Proof.* Let  $\mathbf{f}$  be in the dual space of  $\mathbf{H}_0^{p'}(\operatorname{div}, \Omega)$ . We know that there exist  $\boldsymbol{\psi} \in L^p(\Omega)$  and  $\chi_0 \in L^p(\Omega)$  such that

$$\mathbf{f} = \boldsymbol{\psi} + \nabla \chi_0 \quad \text{and} \quad \|\boldsymbol{\psi}\|_{L^p(\Omega)} + \|\chi_0\|_{L^p(\Omega)} \leq C \|\mathbf{f}\|_{(\mathbf{H}_0^{p'}(\operatorname{div}, \Omega))'}$$

Since

$$\langle \nabla \chi_0, \widetilde{\operatorname{grad}} q_j^T \rangle_{(\mathbf{H}_0^{p'}(\operatorname{div}, \Omega))' \times \mathbf{H}_0^{p'}(\operatorname{div}, \Omega)} = 0$$

and  $\mathbf{f}$  satisfy the compatibility condition (4.9), it is the same for  $\boldsymbol{\psi}$ . Thanks to Theorem 4.8, there exist  $\mathbf{u} \in \mathbf{W}^{2,p}(\Omega)$  and  $\theta \in W^{1,p}(\Omega)$  satisfying

$$\begin{cases} -\Delta \mathbf{u} + \nabla \theta = \boldsymbol{\psi} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} \cdot \mathbf{n} = 0, \operatorname{curl} \mathbf{u} \times \mathbf{n} = \mathbf{0}, & \text{on } \Gamma, \\ \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, \text{ for any } 1 \leq j \leq J, \end{cases}$$

with

$$\|\mathbf{u}\|_{\mathbf{W}^{2,p}(\Omega)} + \|\theta\|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C \|\boldsymbol{\psi}\|_{L^p(\Omega)}.$$

Then,  $\mathbf{u}$  and  $\pi = \theta + \chi_0$  satisfy the announced properties.  $\square$

**REMARK 4.10.** For every  $\mathbf{f}$ ,  $\chi$ ,  $g$ ,  $\mathbf{h}$  with

$$\mathbf{f} \in L^p(\Omega), \chi \in W^{1,p}(\Omega), g \in W^{2-1/p,p}(\Gamma), \mathbf{h} \times \mathbf{n} \in \mathbf{W}^{1-1/p,p}(\Gamma),$$

with the compatibility condition (4.2) and (4.31), the solution  $(\mathbf{u}, \pi)$  of the Stokes problem (4.17) belongs to  $\mathbf{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)/\mathbb{R}$ . Moreover, there exists a constant  $C > 0$  depending only on  $p$  and  $\Omega$  such that:

$$\begin{aligned} \|\mathbf{u}\|_{\mathbf{W}^{2,p}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C & \left( \|\mathbf{f}\|_{L^p(\Omega)} + \|\chi\|_{W^{1,p}(\Omega)} + \|g\|_{W^{2-1/p,p}(\Gamma)} \right. \\ & \left. + \|\mathbf{h} \times \mathbf{n}\|_{\mathbf{W}^{1-1/p,p}(\Gamma)} \right). \end{aligned} \tag{4.22}$$

### 4.3. Very weak solutions

The main new contribution we give in this subsection is the proof of existence for very weak solutions for the Stokes problem  $(\mathcal{S}_T)$ , which extends previous results of [4] for the Dirichlet problem. We consider  $L^p$  solutions with  $1 < p < \infty$  instead of the simpler Hilbert setting and our approach use the results on strong solutions for the Stokes problem  $(\mathcal{S}_T)$ .

With this aim, we introduce the following space:

$$\mathbf{T}^p(\Omega) = \left\{ \boldsymbol{\varphi} \in \mathbf{H}_0^p(\text{div}, \Omega); \text{div } \boldsymbol{\varphi} \in W_0^{1,p}(\Omega) \right\}.$$

Using classical arguments, we can prove the following results.

LEMMA 4.11. *The space  $\mathcal{D}(\Omega)$  is dense in  $\mathbf{T}^p(\Omega)$  and for all  $\chi \in W^{-1,p}(\Omega)$  and  $\boldsymbol{\varphi} \in \mathbf{T}^{p'}(\Omega)$ , we have:*

$$\langle \nabla \chi, \boldsymbol{\varphi} \rangle_{(\mathbf{T}^{p'}(\Omega))' \times \mathbf{T}^{p'}(\Omega)} = - \langle \chi, \text{div } \boldsymbol{\varphi} \rangle_{W^{-1,p}(\Omega) \times W_0^{1,p'}(\Omega)}. \tag{4.23}$$

LEMMA 4.12. *A distribution  $\mathbf{f}$  belongs to  $(\mathbf{T}^p(\Omega))'$  if and only if there exist  $\boldsymbol{\psi} \in L^{p'}(\Omega)$  and  $f_0 \in W^{-1,p'}(\Omega)$ , such that*

$$\mathbf{f} = \boldsymbol{\psi} + \nabla f_0.$$

Moreover, we have the estimate

$$\|\boldsymbol{\psi}\|_{L^{p'}(\Omega)} + \|f_0\|_{W^{-1,p'}(\Omega)} \leq C \|\mathbf{f}\|_{(\mathbf{T}^p(\Omega))'}. \tag{4.24}$$

We shall use the space

$$\mathbf{H}_p(\Delta; \Omega) = \{ \mathbf{v} \in L^p(\Omega); \Delta \mathbf{v} \in (\mathbf{T}^{p'}(\Omega))' \},$$

which is Banach space for the norm:

$$\|\mathbf{v}\|_{\mathbf{H}^p(\Omega)} = \|\mathbf{v}\|_{L^p(\Omega)} + \|\Delta \mathbf{v}\|_{(\mathbf{T}^{p'}(\Omega))'}.$$

The following lemma will help us to prove a trace result.

LEMMA 4.13. *The space  $\mathcal{D}(\overline{\Omega})$  is dense in  $\mathbf{H}_p(\Delta; \Omega)$ .*

We define the space:

$$\mathbf{Y}_T^p(\Omega) = \left\{ \boldsymbol{\varphi} \in \mathbf{W}^{2,p}(\Omega); \boldsymbol{\varphi} \cdot \mathbf{n} = 0, \operatorname{div} \boldsymbol{\varphi} = 0, \operatorname{curl} \boldsymbol{\varphi} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma \right\},$$

and we recall the following relations:

$$\operatorname{div} \mathbf{v} = \operatorname{div}_\Gamma \mathbf{v}_t + K(\mathbf{v} \cdot \mathbf{n}) + \frac{\partial \mathbf{v}}{\partial \mathbf{n}} \cdot \mathbf{n} \text{ on } \Gamma, \tag{4.25}$$

$$\frac{\partial \mathbf{v}}{\partial \mathbf{n}} \cdot \mathbf{n} = \sum_{k=1}^2 v_k \frac{\partial \boldsymbol{\tau}_k}{\partial \mathbf{n}} \cdot \mathbf{n} + \frac{\partial v_n}{\partial \mathbf{n}} \text{ on } \Gamma, \tag{4.26}$$

$$\operatorname{curl} \mathbf{v} = \sum_{j=1}^2 \frac{\partial \mathbf{v}}{\partial s_j} \times \boldsymbol{\tau}_j + \frac{\partial \mathbf{v}}{\partial \mathbf{n}} \times \mathbf{n} \text{ on } \Gamma. \tag{4.27}$$

The following lemma proves that the tangential trace of the **curl** of function  $\mathbf{v}$  of  $\mathbf{H}_p(\Delta; \Omega)$  belongs to  $\mathbf{W}^{-1-\frac{1}{p},p}(\Gamma)$ .

LEMMA 4.14. *The mapping  $\gamma : \mathbf{u} \mapsto \operatorname{curl} \mathbf{u}|_\Gamma \times \mathbf{n}$  on the space  $\mathcal{D}(\overline{\Omega})$  can be extended by continuity to a linear and continuous mapping still denoted by  $\gamma$ , from  $\mathbf{H}_p(\Delta; \Omega)$  into  $\mathbf{W}^{-1-\frac{1}{p},p}(\Gamma)$  and we have the following Green formula: for any  $\mathbf{u} \in \mathbf{H}_p(\Delta; \Omega)$  and  $\boldsymbol{\varphi} \in \mathbf{Y}_T^{p'}(\Omega)$ ,*

$$\langle \Delta \mathbf{u}, \boldsymbol{\varphi} \rangle_{(\mathbf{T}^{p'}(\Omega))' \times \mathbf{T}^{p'}(\Omega)} = \int_\Omega \mathbf{u} \cdot \Delta \boldsymbol{\varphi} \, dx + \langle \operatorname{curl} \mathbf{u} \times \mathbf{n}, \boldsymbol{\varphi} \rangle_\Gamma, \tag{4.28}$$

where the duality on  $\Gamma$  is given by  $\langle \cdot, \cdot \rangle_\Gamma = \langle \cdot, \cdot \rangle_{\mathbf{W}^{-1-\frac{1}{p},p}(\Gamma) \times \mathbf{W}^{1+1/p,p'}(\Gamma)}$ .

*Proof.* Let  $\mathbf{u} \in \mathcal{D}(\overline{\Omega})$ , then formula (4.28) is valid for any  $\boldsymbol{\varphi} \in \mathbf{Y}_T^{p'}(\Omega)$ . Let  $\boldsymbol{\mu} \in \mathbf{W}^{1+1/p,p'}(\Gamma)$  such that  $\boldsymbol{\mu}_n = 0$  on  $\Gamma$ . Then, there exists a function  $\boldsymbol{\varphi} \in \mathbf{W}^{2,p'}(\Omega)$  such that  $\boldsymbol{\varphi}_t = \boldsymbol{\mu}_t$  on  $\Gamma$ . Moreover, using (4.25), we must choose  $\boldsymbol{\varphi}$  such that

$$\frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{n}} \cdot \mathbf{n} = -\operatorname{div}_\Gamma \boldsymbol{\mu}_t \text{ on } \Gamma,$$

in order to obtain  $\operatorname{div} \boldsymbol{\varphi} = 0$  on  $\Gamma$ . We can also fix  $\frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{n}} \times \mathbf{n}$  so that we have

$$\operatorname{curl} \boldsymbol{\varphi} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma.$$

For this, using (4.27) and the fact that  $(\mathbf{z} \times \mathbf{n}) \times \mathbf{n} = -\mathbf{z}_\tau$  for any vector field  $\mathbf{z}$ , we must choose

$$\left( \frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{n}} \right)_\tau = \sum_{j=1}^2 \left( \frac{\partial \boldsymbol{\mu}_t}{\partial s_j} \times \boldsymbol{\tau}_j \right) \times \mathbf{n}.$$

As a consequence,

$$\frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{n}} = -\mathbf{n} \operatorname{div}_{\Gamma} \boldsymbol{\mu}_t + \sum_{j=1}^2 \left( \frac{\partial \boldsymbol{\mu}_t}{\partial s_j} \times \boldsymbol{\tau}_j \right) \times \mathbf{n}$$

satisfies the two conditions:

$$\operatorname{div} \boldsymbol{\varphi} = 0 \text{ and } \operatorname{curl} \boldsymbol{\varphi} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma.$$

Let us summarize. The function  $\boldsymbol{\varphi}$  belongs to  $\mathbf{Y}'_T(\Omega)$  and satisfies:

$$\begin{cases} \boldsymbol{\varphi}_t = \boldsymbol{\mu}_t & \text{on } \Gamma, \\ \frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{n}} = -\mathbf{n} \operatorname{div}_{\Gamma} \boldsymbol{\mu}_t + \sum_{j=1}^2 \left( \frac{\partial \boldsymbol{\mu}_t}{\partial s_j} \times \boldsymbol{\tau}_j \right) \times \mathbf{n} & \text{on } \Gamma, \end{cases} \quad (4.29)$$

such that

$$\|\boldsymbol{\varphi}\|_{\mathbf{W}^{2,p}(\Omega)} \leq C \|\boldsymbol{\mu}\|_{\mathbf{W}^{1+1/p,p'}(\Gamma)}. \quad (4.30)$$

Consequently,

$$\begin{aligned} \left| \langle \operatorname{curl} \mathbf{u} \times \mathbf{n}, \boldsymbol{\mu} \rangle_{\Gamma} \right| &= \left| \langle \operatorname{curl} \mathbf{u} \times \mathbf{n}, \boldsymbol{\varphi}_t \rangle_{\Gamma} \right| \\ &\leq \|\mathbf{u}\|_{\mathbf{L}^p(\Omega)} \|\boldsymbol{\varphi}\|_{\mathbf{W}^{2,p'}(\Omega)} + \|\Delta \mathbf{u}\|_{(\mathbf{T}^{p'}(\Omega))'} \|\boldsymbol{\varphi}\|_{\mathbf{T}^{p'}(\Omega)} \\ &\leq C \|\mathbf{u}\|_{\mathbf{H}_p(\Delta; \Omega)} \|\boldsymbol{\varphi}\|_{\mathbf{W}^{2,p'}(\Omega)}. \end{aligned}$$

Thus, using (4.30), we obtain for any  $\mathbf{u} \in \mathcal{D}(\overline{\Omega})$ :

$$\|\operatorname{curl} \mathbf{u} \times \mathbf{n}\|_{\mathbf{W}^{-1-\frac{1}{p},p}(\Gamma)} \leq C \|\mathbf{u}\|_{\mathbf{H}_p(\Delta; \Omega)}.$$

Therefore, the linear continuous mapping  $\gamma : \mathbf{u} \mapsto \operatorname{curl} \mathbf{u}|_{\Gamma} \times \mathbf{n}$  defined on the space  $\mathcal{D}(\overline{\Omega})$  is continuous for the norm of  $\mathbf{H}_p(\Delta; \Omega)$ . Since  $\mathcal{D}(\overline{\Omega})$  is dense in  $\mathbf{H}_p(\Delta; \Omega)$ , then we can extend this mapping from  $\mathbf{H}_p(\Delta; \Omega)$  into  $\mathbf{W}^{-1-\frac{1}{p},p}(\Gamma)$ .  $\square$

The main result of this subsection is the following.

**THEOREM 4.15.** (Very weak solutions for  $(\mathcal{S}_T)$ ) *Let  $\mathbf{f}$ ,  $\chi$ ,  $g$ , and  $\mathbf{h}$  with*

$$\mathbf{f} \in (\mathbf{T}^{p'}(\Omega))', \chi \in L^p(\Omega), g \in W^{-1/p,p}(\Gamma), \mathbf{h} \times \mathbf{n} \in \mathbf{W}^{-1-1/p,p}(\Gamma),$$

*and satisfying the compatibility conditions (4.2) and*

$$\int_{\Omega} \chi \, d\mathbf{x} = \langle g, 1 \rangle_{\Gamma}. \quad (4.31)$$

*Then, the Stokes problem (4.17) has exactly one solution*

$$\mathbf{u} \in \mathbf{L}^p(\Omega) \text{ and } \pi \in W^{-1,p}(\Omega)/\mathbb{R}.$$

Moreover, there exists a constant  $C > 0$  depending only on  $p$  and  $\Omega$  such that:

$$\begin{aligned} \|\mathbf{u}\|_{L^p(\Omega)} + \|\pi\|_{W^{-1,p}(\Omega)/\mathbb{R}} \leq C \left( \|\mathbf{f}\|_{(\mathbf{T}^{p'}(\Omega))'} + \|\chi\|_{L^p(\Omega)} + \|g\|_{W^{-1/p,p}(\Gamma)} \right. \\ \left. + \|\mathbf{h} \times \mathbf{n}\|_{W^{-1-1/p,p}(\Gamma)} \right). \end{aligned} \tag{4.32}$$

*Proof.* The proof is based on the usual duality argument, which relies on the regularity of the adjoint problem with zero boundary conditions. We proceed in three steps.

THE FIRST STEP: Thanks to Green formula (4.28), it is easy to verify that  $\mathbf{u} \in L^p(\Omega)$  is solution of problem (4.17) without the last condition, is equivalent to the variational formulation: find  $(\mathbf{u}, \pi) \in L^p(\Omega) \times W^{-1,p}(\Omega)$  such that for any  $\boldsymbol{\varphi} \in Y_T^{p'}(\Omega)$ , and for any  $q \in W^{1,p'}(\Omega)$ ,

$$\begin{aligned} - \int_{\Omega} \mathbf{u} \cdot \Delta \boldsymbol{\varphi} \, dx - \langle \pi, \operatorname{div} \boldsymbol{\varphi} \rangle_{W^{-1,p}(\Omega) \times W_0^{1,p'}(\Omega)} = \langle \mathbf{f}, \boldsymbol{\varphi} \rangle_{\Omega} + \langle \mathbf{h} \times \mathbf{n}, \boldsymbol{\varphi} \rangle_{\Gamma} \\ \int_{\Omega} \mathbf{u} \cdot \nabla q \, dx = - \int_{\Omega} \chi q \, dx + \langle g, q \rangle_{W^{-1/p,p}(\Gamma) \times W^{1/p,p'}(\Gamma)}, \end{aligned} \tag{4.33}$$

where the dualities on  $\Omega$  and  $\Gamma$  are defined by:

$$\begin{aligned} \langle \cdot, \cdot \rangle_{\Omega} &= \langle \cdot, \cdot \rangle_{(\mathbf{T}^{p'}(\Omega))' \times \mathbf{T}^{p'}(\Omega)}, \\ \langle \cdot, \cdot \rangle_{\Gamma} &= \langle \cdot, \cdot \rangle_{W^{-1-1/p,p}(\Gamma) \times W^{1+1/p,p'}(\Gamma)}. \end{aligned}$$

Indeed, let  $(\mathbf{u}, \pi) \in L^p(\Omega) \times W^{-1,p}(\Omega)$  be a solution to (4.33). It is clear that:

$$-\Delta \mathbf{u} + \nabla \pi = \mathbf{f} \quad \text{and} \quad \operatorname{div} \mathbf{u} = \chi \text{ in } \Omega.$$

Consequently  $\mathbf{u} \in \mathbf{H}_p(\Delta; \Omega)$ , because the characterization given by Lemma 4.12 implies that  $\nabla \pi \in (\mathbf{T}^{p'}(\Omega))'$ . Using Lemma 4.11 and Lemma 4.14, we obtain for any  $\boldsymbol{\varphi} \in Y_T^{p'}(\Omega)$ :

$$- \int_{\Omega} \mathbf{u} \cdot \Delta \boldsymbol{\varphi} \, dx + \langle \operatorname{curl} \mathbf{u} \times \mathbf{n}, \boldsymbol{\varphi} \rangle_{\Gamma} - \langle \pi, \operatorname{div} \boldsymbol{\varphi} \rangle_{W^{-1,p}(\Omega) \times W_0^{1,p'}(\Omega)} = \langle \mathbf{f}, \boldsymbol{\varphi} \rangle_{\Omega}.$$

Then, we deduce that

$$\langle \operatorname{curl} \mathbf{u} \times \mathbf{n}, \boldsymbol{\varphi} \rangle_{\Gamma} = \langle \mathbf{h} \times \mathbf{n}, \boldsymbol{\varphi} \rangle_{\Gamma}.$$

Let  $\boldsymbol{\mu} \in W^{1+1/p,p'}(\Gamma)$ . As in the proof of Lemma 4.14, there exists a function  $\boldsymbol{\varphi} \in W^{2,p}(\Omega)$  satisfying (4.29). So, we can write that for any  $\boldsymbol{\mu} \in W^{1+1/p,p'}(\Gamma)$ ,

$$\langle \operatorname{curl} \mathbf{u} \times \mathbf{n}, \boldsymbol{\mu} \rangle_{\Gamma} = \langle \mathbf{h} \times \mathbf{n}, \boldsymbol{\mu} \rangle_{\Gamma},$$

which implies that  $\operatorname{curl} \mathbf{u} \times \mathbf{n} = \mathbf{h} \times \mathbf{n}$  on  $\Gamma$ . From the equation  $\operatorname{div} \mathbf{u} = \chi$  in  $\Omega$ , we deduce that for any  $q \in W^{1,p'}(\Omega)$ , we have

$$\langle \mathbf{u} \cdot \mathbf{n}, q \rangle_{W^{-1/p,p}(\Gamma) \times W^{1/p,p'}(\Gamma)} = \langle g, q \rangle_{W^{-1/p,p}(\Gamma) \times W^{1/p,p'}(\Gamma)}.$$

Consequently,  $\mathbf{u} \cdot \mathbf{n} = g$  in  $W^{-1/p,p}(\Gamma)$ . The converse is a simple consequence of Lemma 4.11, Lemma 4.13 and Lemma 4.14.

THE SECOND STEP: Let's now solve problem (4.33). We suppose that

$$g = 0 \text{ on } \Gamma \text{ and } \int_{\Omega} \chi \, d\mathbf{x} = 0.$$

We know due to Corollary 4.10 that for any pair

$$(\mathbf{F}, \xi) \in (\mathbf{L}^{p'}(\Omega) \perp \mathbf{K}_T^p(\Omega)) \times (W_0^{1,p'}(\Omega) \cap L_0^{p'}(\Omega)),$$

there exists a unique  $\boldsymbol{\varphi} \in \mathbf{W}^{2,p'}(\Omega)$  and  $q \in W^{1,p'}(\Omega)/\mathbb{R}$  satisfying:

$$\begin{cases} -\Delta \boldsymbol{\varphi} + \nabla q = \mathbf{F} & \text{and } \operatorname{div} \boldsymbol{\varphi} = \xi & \text{in } \Omega, \\ \boldsymbol{\varphi} \cdot \mathbf{n} = 0 \text{ and } \operatorname{curl} \boldsymbol{\varphi} \times \mathbf{n} = \mathbf{0} & & \text{on } \Gamma, \\ \langle \boldsymbol{\varphi} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, & \text{for any } 1 \leq j \leq J, \end{cases}$$

with the estimate

$$\|\boldsymbol{\varphi}\|_{\mathbf{W}^{2,p'}(\Omega)} + \|q\|_{W^{1,p'}(\Omega)/\mathbb{R}} \leq C \{ \|\mathbf{F}\|_{\mathbf{L}^{p'}(\Omega)} + \|\xi\|_{W^{1,p'}(\Omega)} \}. \tag{4.34}$$

Note that for any  $K \in \mathbb{R}$ ,

$$\left| \int_{\Omega} \chi q \, d\mathbf{x} \right| = \left| \int_{\Omega} \chi (q + K) \, d\mathbf{x} \right| \leq \|\chi\|_{L^p(\Omega)} \|q\|_{L^{p'}(\Omega)/\mathbb{R}}$$

and

$$|\langle \mathbf{f}, \boldsymbol{\varphi} \rangle_{\Omega}| \leq \|\mathbf{f}\|_{(\mathbf{T}^{p'}(\Omega))'} \|\boldsymbol{\varphi}\|_{\mathbf{T}^{p'}(\Omega)} \leq \|\mathbf{f}\|_{(\mathbf{T}^{p'}(\Omega))'} \|\boldsymbol{\varphi}\|_{\mathbf{W}^{2,p'}(\Omega)}.$$

From these bounds, we have

$$\begin{aligned} & \left| \langle \mathbf{f}, \boldsymbol{\varphi} \rangle_{\Omega} + \langle \mathbf{h} \times \mathbf{n}, \boldsymbol{\varphi} \rangle_{\Gamma} - \int_{\Omega} \chi (q + K) \, d\mathbf{x} \right| \\ & \leq \left( \|\mathbf{f}\|_{(\mathbf{N}^{p'}(\Omega))'} + \|\mathbf{h} \times \mathbf{n}\|_{\mathbf{W}^{-1-1/p,p}(\Gamma)} + \|\chi\|_{L^p(\Omega)} \right) \\ & \quad \times \left( \|\mathbf{F}\|_{\mathbf{L}^{p'}(\Omega)} + \|\xi\|_{W^{1,p'}(\Omega)} \right). \end{aligned}$$

In other words, we can say that the linear mapping:

$$(\mathbf{F}, \xi) \mapsto \langle \mathbf{f}, \boldsymbol{\varphi} \rangle_{\Omega} + \langle \mathbf{h} \times \mathbf{n}, \boldsymbol{\varphi} \rangle_{\Gamma} - \int_{\Omega} \chi q \, d\mathbf{x}$$

defines an element of the dual space of

$$(\mathbf{L}^{p'}(\Omega) \perp \mathbf{K}_T^p(\Omega)) \times (W_0^{1,p'}(\Omega) \cap L_0^{p'}(\Omega))$$

and according to the Riesz’s representation theorem, there exists a unique

$$(\mathbf{u}, \pi) \in (\mathbf{L}^p(\Omega)/\mathbf{K}_T^p(\Omega)) \times W^{-1,p}(\Omega)/\mathbb{R}$$

solution of problem (4.33) satisfying the bound (4.32).

A such solution  $(\mathbf{u}, \pi)$  satisfies the problem  $(\mathcal{S}_T)$  without the last condition but we have only to set

$$\tilde{\mathbf{u}} = \mathbf{u} - \sum_{i=1}^I \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} \widetilde{\mathbf{grad}} q_j^T.$$

It is clear that  $(\tilde{\mathbf{u}}, \pi) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$  is also solution of  $(\mathcal{S}_T)$  and satisfies its last condition.

THE THIRD STEP: Now, we suppose that  $g \neq 0$  and the compatibility condition (4.31) holds. We consider the Neumann problem:

$$\Delta \theta = \chi \text{ in } \Omega \quad \text{and} \quad \frac{\partial \theta}{\partial \mathbf{n}} = g \text{ on } \Gamma,$$

which has a unique solution  $\theta \in W^{1,p}(\Omega)/\mathbb{R}$  satisfying the estimate:

$$\|\theta\|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C(\|\chi\|_{L^p(\Omega)} + \|g\|_{W^{-1/p,p}(\Gamma)}). \tag{4.35}$$

Set  $\mathbf{w} = \nabla \theta$ . By step i), there exists a unique  $(\mathbf{z}, \pi) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R}$  solution of problem:

$$\begin{cases} -\Delta \mathbf{z} + \nabla \pi = \mathbf{f} + \Delta \mathbf{w} & \text{in } \Omega, \\ \operatorname{div} \mathbf{z} = 0 & \text{in } \Omega, \\ \mathbf{z} \cdot \mathbf{n} = 0 \text{ and } \operatorname{curl} \mathbf{z} \times \mathbf{n} = \mathbf{0} & \text{on } \Gamma, \\ \langle \mathbf{z} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, \text{ for any } 1 \leq j \leq J, \end{cases}$$

where  $\Delta \mathbf{w} = \nabla \chi$  and the characterization given by Lemma 4.12 implies that  $\Delta \mathbf{w} \in (\mathbf{N}^{p'}(\Omega))'$ . Finally, the pair of functions  $(\mathbf{u}, \pi) = (\mathbf{z} + \mathbf{w}, \pi)$  is the required solution.  $\square$

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