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OSCILLATION CRITERIA OF CERTAIN THIRD ORDER NEUTRAL DIFFERENTIAL EQUATIONS

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Abstract. This paper deals with the oscillation of the following third order neutral delay differential equation

$$(r(t)|z''(t)|^{\alpha-1}z''(t))' + q(t)|x(\tau(t))|^{\alpha-1}x(\tau(t)) = 0,$$

where $t \geqslant t_0$, $\alpha > 0$ and $z(t) = x(t) + p(t)x(\delta(t))$. We will establish some new sufficient conditions which insure that any solution of this equation oscillates or converges to zero. Two examples are also provided to illustrate the relevance of the main results.

1. Introduction

Consider the third order neutral delay differential equation

$$(r(t)|z''(t)|^{\alpha-1}z''(t))' + q(t)|x(\tau(t))|^{\alpha-1}x(\tau(t)) = 0, \tag{1.1}$$

where $t \geqslant t_0$, $\alpha > 0$ is a fixed constant and $z(t) = x(t) + p(t)x(\delta(t))$. Throughout this paper, we assume that:

(A1) $p(t), q(t) \in C([t_0,\infty),\mathbb{R})$ with $-\mu \leqslant p(t) \leqslant p < 1$ for $\mu \in (0,1)$ and $q(t) \geqslant 0$;

(A2) $r(t) \in C^1([t_0,\infty),(0,\infty))$ with $r'(t) \geqslant 0$ and $\int_{t_0}^{\infty} r^{-1/\alpha}(s)ds = \infty$;

(A3) $\delta(t), \tau(t) \in C([t_0,\infty),\mathbb{R})$ with $\delta(t) \leqslant t$, $\tau(t) \leqslant t$, $\lim_{t\to\infty}\delta(t) = \lim_{t\to\infty}\tau(t) = \infty$.

By a solution of Eq.(1.1) we mean a function $x(t) \in C^1([T_x,\infty),\mathbb{R})$, $T_x \geqslant t_0$, which has the property $r(t)|z''(t)|^{\alpha-1}z''(t) \in C^1([T_x,\infty),\mathbb{R})$ and satisfies Eq.(1.1) for all $t \geqslant T_x$. We consider only those solutions $x(t)$ of Eq.(1.1) which satisfy

$$\sup\{|x(t)| : t \geqslant T\} > 0 \text{ for all } t \geqslant T_x.$$

We assume that Eq.(1.1) possesses such a solution [13]. As is customary, a solution of Eq.(1.1) is said to be oscillatory if it has arbitrarily large zeros. Otherwise, it is said to be nonoscillatory.

Neutral delay differential equations have applications to electric networks containing lossless transmission lines. Such networks appear in high speed computers where

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lossless transmission less are used to interconnect switching circuits. They also occur in problems dealing with vibrating masses attached to elastic bar and in some variational problems, see [13].

The problem of the oscillation of solutions of differential equations has been widely studied by many authors and by many techniques since the pioneering work of Sturm on second order linear differential equations. In the past 30 years, the oscillation theory for second order neutral delay differential equations and third-order retarded delay differential equations have been well developed; see, for example, the monographs [4, 10] and papers [1, 2, 5, 7, 8, 9, 11, 15] as well as the references cites therein. Compared to second order neutral delay differential equations, it seems that not much work has been done concerning with the oscillation and asymptotic of third order neutral differential equations [3, 12]. Very recently, Baculíková and Džurina [3] have studied the oscillation behavior of Eq.(1.1) and extended Nehari's theorems [14] to Eq.(1.1).

Motivated by the recent works [3, 6], in this paper, we will establish some new sufficient conditions which insure that any solution of Eq.(1.1) oscillates or converges to zero. The theorems obtained here extend the main theorems [6] to Eq.(1.1) and complement the existing results in [3]. Finally, two examples are also provided to illustrate the relevance of the main results.

2. Oscillation criteria for $0 \leqslant p(t) \leqslant p < 1$

In this section, we will establish some oscillation criteria for Eq.(1.1) in the case when $0 \leqslant p(t) \leqslant p < 1$. In order to prove our main results, we need the following lemmas.

LEMMA 2.1. (see [3, Lemma 1]) *Let $x(t)$ be a positive solution of Eq.(1.1). Then there are only the following two cases for $z(t)$:*

(i) $z(t) > 0,\ z'(t) > 0,\ z''(t) > 0$;

(ii) $z(t) > 0,\ z'(t) < 0,\ z''(t) > 0$,

for $t \geqslant t_1$, where t_1 is sufficiently large.

LEMMA 2.2. (see [3, Lemma 2]) *Let $x(t)$ be a positive solution of Eq.(1.1) and $z(t)$ satisfy Lemma 2.1 (ii). If*

$$\int_{t_0}^{\infty}\int_{v}^{\infty}\left[\frac{1}{r(u)}\int_{u}^{\infty} q(s)ds\right]^{1/\alpha} dudv = \infty, \tag{2.1}$$

then $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} z(t) = 0$.

LEMMA 2.3. (see [3, Lemma 3]) *Let $u(t) \in C^2([t_0,\infty),\mathbb{R})$. Assume that $u(t) > 0$, $u'(t) \geqslant 0$, and $u''(t) \leqslant 0$ on $[t_0,\infty)$. Then for each $k_1 \in (0,1)$ there exist a $T_1 \geqslant t_0$ such that*

$$\frac{u(\tau(t))}{u(t)} \geqslant k_1 \frac{\tau(t)}{t},\ \ t \geqslant T_1.$$

LEMMA 2.4. *Let $z(t) \in C^3([t_0,\infty),\mathbb{R})$. Assume that $z(t)>0$, $z'(t)>0$, $z''(t)>0$, $z'''(t) \leqslant 0$ on $[t_0,\infty)$. Then for each $k_2 \in (0,1)$ there exist a $T_2 \geqslant t_0$ such that*

$$z(t) \geqslant \frac{1}{2}k_2 t z'(t), \ \ t \geqslant T_2.$$

The proof of this lemma proceeds along the lines of that of [3, Lemma 4] and hence is omitted.

For simplicity, for each k_1, $k_2 \in (0,1)$, define

$$Q(t) = \Big(\frac{k_1 k_2}{2}\Big)^{\alpha}\Big(\frac{\tau^2(t)}{t}\Big)^{\alpha}[1-p(\tau(t))]^{\alpha}q(t), \ t \geqslant t_0.$$

LEMMA 2.5. *Let $x(t)$ be a positive solution of Eq.(1.1) and $z(t)$ satisfy Lemma 2.1(i). Then for each $k_1,k_2 \in (0,1)$, there exist a $T_3 \in [t_0,\infty)$ and a positive function $w(t)$ defined on $[T_3,\infty)$ such that for $t \geqslant T_3$,*

$$\int_t^{\infty} Q(s)ds < \infty, \quad \int_t^{\infty}\Big(\frac{w^{\alpha+1}(s)}{r(s)}\Big)^{1/\alpha}ds < \infty, \tag{2.2}$$

and

$$w(t) \geqslant \int_t^{\infty} Q(s)ds + \alpha\int_t^{\infty}\Big(\frac{w^{\alpha+1}(s)}{r(s)}\Big)^{1/\alpha}ds. \tag{2.3}$$

Proof. Without loss of generality, we assume that $x(t)>0$, $x(\delta(t))>0$, $x(\tau(t))>0$ for $t \geqslant t_2 \geqslant t_0$ and $z(t)$ satisfies Lemma 2.1 (i). Note that

$$x(t) = z(t) - p(t)x(\delta(t)) > z(t) - p(t)z(\delta(t)) \geqslant (1-p(t))z(t),$$

which follows from (1.1) that

$$(r(t)|z''(t)|^{\alpha-1}z''(t))' \leqslant -q(t)[1-p(\tau(t))]^{\alpha}z^{\alpha}(\tau(t)), \tag{2.4}$$

so,

$$(r(t)|z''(t)|^{\alpha-1}z''(t))' \leqslant 0.$$

The last inequality together with $r'(t) \geqslant 0$ and $z''(t)>0$ gives $z'''(t) \leqslant 0$. So there exists a $t_3 \geqslant t_2$ such that $z(t)$ satisfies:

$$z(\tau(t))>0,\ z'(t)>0,\ z''(t)>0 \ \text{ and } \ z'''(t) \leqslant 0, \quad t \geqslant t_3.$$

Define

$$w(t) = r(t)\Big(\frac{z''(t)}{z'(t)}\Big)^{\alpha}, \ t \geqslant t_3. \tag{2.5}$$

Obviously, $w(t)>0$. By (2.4),

$$\begin{aligned} w'(t) &= \frac{(r(t)(z''(t))^{\alpha})'}{(z'(t))^{\alpha}} - \alpha r(t)\Big(\frac{z''(t)}{z'(t)}\Big)^{\alpha+1} \\ &\leqslant -q(t)[1-p(\tau(t))]^{\alpha}\Big(\frac{z(\tau(t))}{z'(t)}\Big)^{\alpha} - \alpha\Big(\frac{w^{\alpha+1}(t)}{r(t)}\Big)^{1/\alpha}. \end{aligned} \tag{2.6}$$

From Lemma 2.3 with $u(t) = z'(t)$, for k_1 the same as in $Q(t)$,

$$\frac{1}{z'(t)} \geqslant k_1 \frac{\tau(t)}{t} \frac{1}{z'(\tau(t))}, \quad t \geqslant T_1 \geqslant t_2, \tag{2.7}$$

and by Lemma 2.4, for k_2 the same as in $Q(t)$, by (2.7),

$$\frac{z(\tau(t))}{z'(t)} \geqslant k_1 \frac{\tau(t)}{t} \frac{z(\tau(t))}{z'(\tau(t))} \geqslant \frac{k_1 k_2}{2} \frac{\tau^2(t)}{t}, \ t \geqslant t_4 \geqslant t_3. \tag{2.8}$$

Combining (2.6) and (2.8), we get

$$w'(t) \leqslant -Q(t) - \alpha \Big(\frac{w^{\alpha+1}(t)}{r(t)} \Big)^{1/\alpha}, \ t \geqslant t_4.$$

Integrating the above inequality from t to T, $T \geqslant t_4$, we obtain

$$w(T) - w(t) + \int_t^T Q(s)ds + \alpha \int_t^T \Big(\frac{w^{\alpha+1}(s)}{r(s)} \Big)^{1/\alpha} ds \leqslant 0. \tag{2.9}$$

We now claim that $\int_t^\infty Q(s)ds < \infty$. Otherwise, it follows from (2.9) that

$$w(T) \leqslant w(t) - \int_t^T Q(s)ds \to -\infty \quad \text{as } T \to \infty,$$

which is a contradiction. Hence the claim is proved. Similarly, we can show

$$\int_t^\infty \Big(\frac{w^{\alpha+1}(s)}{r(s)} \Big)^{1/\alpha} ds < \infty, \ t \geqslant t_4. \tag{2.10}$$

By (2.6), $\lim_{t\to\infty} w(t) = w^* \geqslant 0$ exists. In view of (2.10), we have $w^* = 0$. Letting $T \to \infty$ in (2.9), (2.3) holds. This completes the proof. □

Define a sequence of functions $\{A_n(t)\}_{n=0}^\infty$ as

$$A_0(t) = \int_t^\infty Q(s)ds, \ \ t \geqslant t_0,$$

and

$$A_n(t) = \alpha \int_t^\infty \Big(\frac{A_{n-1}^{\alpha+1}(s)}{r(s)} \Big)^{1/\alpha} ds + A_0(t), \ t \geqslant t_0, \ \ n = 1, 2, \cdots. \tag{2.11}$$

In induction method, it is easy to prove that (2.11) is a nondecreasing, i.e.,

$$A_n(t) \leqslant A_{n+1}(t), \quad t \geqslant t_0, \ \ n = 1, 2, \cdots. \tag{2.12}$$

LEMMA 2.6. *Let $x(t)$ be a positive solution of Eq.(1.1) and $z(t)$ satisfy Lemma 2.1(i). Then there exist a $T_4 \in [t_0, \infty)$ and a positive function $A(t)$ defined on $[T_4, \infty)$ such that $\lim_{n\to\infty} A_n(t) = A(t)$ for $t \geqslant T_4$. Furthermore,*

$$A(t) = \alpha \int_t^\infty \Big(\frac{A^{\alpha+1}(s)}{r(s)} \Big)^{1/\alpha} ds + A_0(t), \ t \geqslant T_4. \tag{2.13}$$

Proof. By Lemma 2.5, for each k_1, $k_2 \in (0,1)$, there exist a $T_3 \in [t_0,\infty)$ and a positive function $w(t)$ such that (2.3) holds. Consequently, $w(t) \geqslant A_0(t)$ for $t \geqslant T_3$. Inductively, we can get $w(t) \geqslant A_n(t)$ for $t \geqslant T_3$, $n = 1,2,\cdots$. Thus, by (2.12), the sequence $\{A_n(t)\}_{n=0}^{\infty}$ converges to $A(t)$ on $[T_3,\infty)$. By Lebesgue monotone convergence theorem and letting $n \to \infty$ in (2.11), (2.13) holds. □

Now we present our main oscillation results for Eq.(1.1)

THEOREM 2.1. *Assume that (2.1) holds. If* $\int_{t_0}^{\infty} Q(s)ds = \infty$, *then any solution* $x(t)$ *of Eq.(1.1) is oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof. Suppose that $x(t)$ is a nonoscillatory solution of Eq.(1.1) on $[t_0,\infty)$. Without loss of generality, we may assume that $x(t)$ is a positive solution of Eq.(1.1). In view of Lemma 2.1, $z(t)$ only satisfies Lemma 2.1 (i) or (ii).

Assume that $z(t)$ satisfies Lemma 2.1 (i). It follows from Lemma 2.5 that

$$\int_{t_0}^{\infty} Q(s)ds < \infty,$$

which is a contradiction.

Next we assume that $z(t)$ satisfies Lemma 2.1 (ii). Note that (2.1) holds, by Lemma 2.2, $\lim_{t\to\infty} x(t) = 0$, Hence, we complete the proof. □

THEOREM 2.2. *Assume that (2.1) holds. If*

$$\liminf_{t\to\infty} \frac{1}{A_0(t)} \int_t^{\infty} \left(\frac{A_0^{\alpha+1}(s)}{r(s)}\right)^{1/\alpha} ds > \frac{1}{(\alpha+1)^{(\alpha+1)/\alpha}}, \tag{2.14}$$

then any solution $x(t)$ *of Eq.(1.1) is oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof. Suppose that $x(t)$ is a nonoscillatory solution of Eq.(1.1) on $[t_0,\infty)$. Without loss of generality, we assume that $x(t)$ is a positive solution of Eq.(1.1). In view of Lemma 2.1, $z(t)$ only satisfies Lemma 2.1 (i) or (ii).

If $z(t)$ satisfies Lemma 2.1 (ii), then from Lemma 2.2 follows $\lim_{t\to\infty} x(t) = 0$.

Next, we assume that $z(t)$ satisfies Lemma 2.1 (i), and let $w(t)$ be defined by (2.5). Then it follows from Lemma 2.5 that (2.3) holds. By (2.14), there exists a constant $\eta > (\alpha+1)^{-(\alpha+1)/\alpha}$ such that

$$\liminf_{t\to\infty} \frac{1}{A_0(t)} \int_t^{\infty} \left(\frac{A_0^{\alpha+1}(s)}{r(s)}\right)^{1/\alpha} ds > \eta. \tag{2.15}$$

On the other hand, by (2.3),

$$\begin{aligned} \frac{w(t)}{A_0(t)} &\geqslant 1 + \frac{\alpha}{A_0(t)} \int_t^{\infty} \left(\frac{w^{\alpha+1}(s)}{r(s)}\right)^{1/\alpha} ds \\ &= 1 + \frac{\alpha}{A_0(t)} \int_t^{\infty} \left(\frac{A_0^{\alpha+1}(s)}{r(s)}\right)^{1/\alpha} \left(\frac{w(s)}{A_0(s)}\right)^{(\alpha+1)/\alpha} ds, \ t \geqslant t_1. \end{aligned} \tag{2.16}$$

Let $\lambda = \inf_{t \geqslant t_1} \frac{w(t)}{A_0(t)}$, then $\lambda \geqslant 1$. It follows from (2.15) and (2.16) that

$$\lambda \geqslant 1 + \alpha \lambda^{(\alpha+1)/\alpha} \eta \geqslant 1 + \alpha \Big(\frac{\lambda}{\alpha+1} \Big)^{(\alpha+1)/\alpha},$$

i.e.,

$$\frac{\lambda}{\alpha+1} \geqslant \frac{1}{\alpha+1} + \frac{\alpha}{\alpha+1} \Big(\frac{\lambda}{\alpha+1} \Big)^{(\alpha+1)/\alpha},$$

which contradicts the admissible values of η and λ. It completes the proof. □

Denote

$$Q_0(t) = \int_t^\infty \Big(\frac{\tau^2(s)}{s} \Big)^\alpha [1 - p(\tau(s))]^\alpha q(s) ds, \ t \geqslant t_0.$$

By Theorem 2.2, we get the following sharp result.

COROLLARY 2.1. *Assume that (2.1) holds. If*

$$\liminf_{t \to \infty} \frac{1}{Q_0(t)} \int_t^\infty \Big(\frac{Q_0^{\alpha+1}(s)}{r(s)} \Big)^{1/\alpha} ds > \frac{2}{(\alpha+1)^{(\alpha+1)/\alpha}}, \tag{2.17}$$

then any solution $x(t)$ of Eq.(1.1) is oscillatory or satisfies $\lim_{t \to \infty} x(t) = 0$.

Proof. We shall show that (2.17) implies (2.14). Note that

$$\frac{1}{A_0(t)} \int_t^\infty \Big(\frac{A_0^{\alpha+1}(s)}{r(s)} \Big)^{1/\alpha} ds = \frac{k'}{2Q_0(t)} \int_t^\infty \Big(\frac{Q_0^{\alpha+1}(s)}{r(s)} \Big)^{1/\alpha} ds, \tag{2.18}$$

where $k' = k_1 k_2$. On the other hand, (2.17) implies that for some $k' \in (0,1)$,

$$\liminf_{t \to \infty} \frac{1}{Q_0(t)} \int_t^\infty \Big(\frac{Q_0^{\alpha+1}(s)}{r(s)} \Big)^{1/\alpha} ds > \frac{1}{k'} \frac{2}{(\alpha+1)^{(\alpha+1)/\alpha}}. \tag{2.19}$$

Combining (2.18) with (2.19), we get (2.14) holds. Hence, by Theorem 2.2, we complete the proof. □

THEOREM 2.3. *Assume that (2.1) holds. If there is a positive integer m such that either of the following conditions hold:*

$$\int_{t_0}^\infty Q(t) \exp\Big(\alpha \int_{t_0}^t \Big(\frac{A_m(s)}{r(s)} \Big)^{1/\alpha} ds \Big) dt = \infty, \tag{2.20}$$

or

$$\int_{t_0}^\infty \Big(\frac{A_m(t)}{r(t)} \Big)^{1/\alpha} A_0(t) \exp\Big(\alpha \int_{t_0}^t \Big(\frac{A_m(s)}{r(s)} \Big)^{1/\alpha} ds \Big) dt = \infty, \tag{2.21}$$

then any solution $x(t)$ of Eq.(1.1) is oscillatory or satisfies $\lim_{t \to \infty} x(t) = 0$.

Proof. Suppose that $x(t)$ is a nonoscillatory solution of Eq.(1.1) on $[t_0,\infty)$. Without loss of generality, we assume that $x(t)$ is a positive solution of Eq.(1.1). In view of Lemma 2.1, $z(t)$ only satisfies Lemma 2.1 (i) or (ii).

If $z(t)$ satisfies Lemma 2.1 (ii), then, by Lemma 2.2, we have $\lim_{t\to\infty} x(t)=0$.

Assume that $z(t)$ satisfies Lemma2.1 (i). By Lemma 2.6, (2.13) holds. Thus,

$$\begin{aligned} A'(t) &= -\alpha\Big(\frac{A^{\alpha+1}(t)}{r(t)}\Big)^{1/\alpha} - Q(t) \\ &\leqslant -\alpha\Big(\frac{A_m(t)}{r(t)}\Big)^{1/\alpha} A(t) - Q(t),\ t \geqslant T_4, \end{aligned}$$

since $A_m(t)\leqslant A(t)$. The above inequality follows

$$\begin{aligned} A(t) \leqslant \exp\Big(-\alpha\int_{T_4}^{t}\Big(\frac{A_m(s)}{r(s)}\Big)^{1/\alpha} ds\Big) \\ \times\Big[A(T_4) - \int_{T_4}^{t} Q(s)\exp\Big(\alpha\int_{T_4}^{s}\Big(\frac{A_m(u)}{r(u)}\Big)^{1/\alpha} du\Big)ds\Big], \end{aligned}$$

so,

$$\infty > A(T_4) \geqslant \int_{T_4}^{t} Q(s)\exp\Big(\alpha\int_{T_4}^{s}\Big(\frac{A_m(u)}{r(u)}\Big)^{1/\alpha} du\Big)ds,$$

which contradicts (2.20).

On the other hand, define

$$v(t) = \alpha\int_{t}^{\infty}\Big(\frac{A^{\alpha+1}(s)}{r(s)}\Big)^{1/\alpha} ds,\ t\geqslant T_4.$$

Hence, by (2.13),

$$v'(t) = -\alpha\Big(\frac{A^{\alpha+1}(t)}{r(t)}\Big)^{1/\alpha} \leqslant -\alpha\Big(\frac{A_m(t)}{r(t)}\Big)^{1/\alpha} A(t) = -\alpha\Big(\frac{A_m(t)}{r(t)}\Big)^{1/\alpha}[v(t)+A_0(t)].$$

Similarly, we get

$$\int_{T_3}^{\infty}\Big(\frac{A_m(t)}{r(t)}\Big)^{1/\alpha} A_0(t)\exp\Big(\alpha\int_{T_3}^{t}\Big(\frac{A_m(s)}{r(s)}\Big)^{1/\alpha} ds\Big)dt < \infty,$$

which contradicts (2.21). This completes the proof. □

EXAMPLE 2.1. Consider the third order differential equation

$$\Big(t|z''(t)|^3 z''(t)\Big)' + \frac{c_1}{t^8}\Big|x\Big(\frac{t}{3}\Big)\Big|^3 x\Big(\frac{t}{3}\Big) = 0, \quad t\geqslant 1, \tag{2.22}$$

where $z(t)=x(t)+\frac{1}{2}x(\frac{t}{2})$, $c_1\geqslant 0$. For Eq.(1.1), let

$$\alpha=4,\ r(t)=t,\ q(t)=\frac{c_1}{t^8},\ p(t)=\frac{1}{2},\ \tau(t)=\frac{t}{3},\ \delta(t)=\frac{t}{2}.$$

Note that

$$\int_1^\infty \int_v^\infty \Big[\frac{1}{r(u)}\int_u^\infty q(s)ds\Big]^{1/4} dudv = \big(\frac{c_1}{7}\big)^{1/4}\int_{t_0}^\infty \frac{1}{v}dv = \infty,$$

i.e., (2.1) holds. Here,

$$Q_0(t) = \int_t^\infty \Big(\frac{\tau^2(s)}{s}\Big)^4 (1-p(\tau(s)))^4 q(s)ds = \frac{c_1}{2^4 3^9 t^3}.$$

So,

$$\liminf_{t\to\infty} \frac{1}{Q_0(t)}\int_t^\infty \Big(\frac{Q_0^5(s)}{r(s)}\Big)^{1/4} ds = \frac{c_1^{1/4}}{2\cdot 3^{13/4}}.$$

Hence, by Corollary 2.1, every nonoscillatory solution of Eq.(2.22) converges to zero provided that $c_1 > 2^8 3^{13}/5^5$. In fact, let $c_1 = 176$, one such solution is $x(t) = 1/t$.

3. Oscillation criteria for $-\mu \leqslant p(t) \leqslant 0$

Similar to Section 2, in this section we will present some oscillation criteria for Eq.(1.1) under the case when $-\mu \leqslant p(t) \leqslant 0$ for $\mu \in (0,1)$. In order to prove the main results, we need the following Lemmas.

LEMMA 3.1. (see [3, Lemma 7]) *Let $x(t)$ be a positive solution of Eq.(1.1). Then there are only the following four cases for $z(t)$:*

(j) $z(t) > 0, z'(t) > 0, z''(t) > 0$;

(jj) $z(t) > 0, z'(t) < 0, z''(t) > 0$;

(jjj) $z(t) < 0, z'(t) < 0, z''(t) > 0$;

(jv) $z(t) < 0, z'(t) < 0, z''(t) < 0$,

for $t \geqslant t_1$, where t_1 is sufficiently large.

LEMMA 3.2. (see [3, Lemma 8]) *Let $x(t)$ be a positive solution of Eq.(1.1) and $z(t)$ satisfy Lemma 3.1 (jj). If (2.1) holds, then* $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} z(t) = 0$.

For simplicity, for each k_1, $k_2 \in (0,1)$, define

$$\widetilde{Q}(t) = \Big(\frac{k_1 k_2}{2}\Big)^\alpha \Big(\frac{\tau^2(t)}{t}\Big)^\alpha q(t),\ t \geqslant t_0.$$

LEMMA 3.3. *Let $x(t)$ be a positive solution of Eq.(1.1) and $z(t)$ satisfy Lemma 3.1 (j). Then for each $k_1, k_2 \in (0,1)$, there exist a $T_5 \in [t_0,\infty)$ and a positive function $w(t)$ defined on $[T_5,\infty)$ such that for $t \geqslant T_5$,*

$$\int_t^\infty \widetilde{Q}(s)ds < \infty, \quad \int_t^\infty \Big(\frac{w^{\alpha+1}(s)}{r(s)}\Big)^{1/\alpha} ds < \infty, \tag{3.1}$$

and

$$w(t) \geqslant \int_t^\infty \widetilde{Q}(s)ds + \alpha \int_t^\infty \left(\frac{w^{\alpha+1}(s)}{r(s)}\right)^{1/\alpha} ds. \tag{3.2}$$

Proof. Without loss of generality, we assume that $x(t)$ is a positive solution of Eq.(1.1) and $z(t)$ satisfied Lemma 3.1 (j). Noting that $-\mu \leqslant p(t) \leqslant 0$, we have $x(t) \geqslant z(t)$. Then, it follows from Eq.(1.1) that

$$(r(t)(z''(t))^\alpha)' = -q(t)x^\alpha(\tau(t)) \leqslant -q(t)z^\alpha(\tau(t)).$$

Replacing Q(t) by $\widetilde{Q}(t)$ and following the similar steps as in the proof of Lemma 2.5, we can get all desired results. □

Similar to (2.11), define a sequence of functions $\{\widetilde{A}_n(t)\}_{n=0}^\infty$ as

$$\widetilde{A}_0(t) = \int_t^\infty \widetilde{Q}(s)ds, \ t \geqslant t_0,$$

and

$$\widetilde{A}_n(t) = \alpha \int_t^\infty \left(\frac{\widetilde{A}_{n-1}^{\alpha+1}(s)}{r(s)}\right)^{1/\alpha} ds + \widetilde{A}_0(t), \quad t \geqslant t_0, \ n = 1, 2, \cdots. \tag{3.3}$$

Proceeding as the proof of Lemma 2.6, we have

LEMMA 3.4. *Let $x(t)$ be a positive solution of Eq.(1.1) and $z(t)$ satisfy Lemma 3.1 (j). Then there exist a $T_6 \in [t_0, \infty)$ and a positive function $\widetilde{A}(t)$ defined on $[T_6, \infty)$ such that $\lim_{n\to\infty} \widetilde{A}_n(t) = \widetilde{A}(t)$ for $t \geqslant T_6$. Furthermore,*

$$\widetilde{A}(t) = \alpha \int_t^\infty \left(\frac{\widetilde{A}^{\alpha+1}(s)}{r(s)}\right)^{1/\alpha} ds + \widetilde{A}_0(t), \ t \geqslant T_6. \tag{3.4}$$

THEOREM 3.1. *Assume that (2.1) holds. If $\int_{t_0}^\infty \widetilde{Q}(s)ds = \infty$, then any solution $x(t)$ of Eq.(1.1) is oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.*

Proof. Assume that $x(t)$ be a nonoscillatory solution of Eq.(1.1). Without loss of generality, we assume that $x(t)$ is a positive solution of Eq.(1.1).

We claim that $x(t)$ is bounded. To prove this we assume, on the contrary, that $x(t)$ is unbounded. Hence there exists a sequence t_m such that $\lim_{m\to\infty} t_m = \infty$; moreover $\lim_{m\to\infty} x(t_m) = \infty$ and

$$x(t_m) = \max\{x(s); t_1 \leqslant s \leqslant t_m\}, \ m \geqslant 2.$$

Since $\lim_{t\to\infty} \delta(t) = \infty$, we can choose m sufficiently large that $\delta(t_m) > t_2$. Noting that $\delta(t) \leqslant t$, we have

$$x(\delta(t_m)) = \max\{x(s); t_1 \leqslant s \leqslant \delta(t_m)\} \leqslant \max\{x(s); t_1 \leqslant s \leqslant t_m\} = x(t_m).$$

Therefore, for all large m,

$$z(t_m) = x(t_m) + p(t_m)x(\delta(t_m)) \geqslant (1-\mu)x(t_m).$$

Thus $z(t_m) \to \infty$ as $m \to \infty$. So, $z(t) > 0$ is positive and unbounded. It follows from Lemma 3.1 that Lemma 3.1 (j) has to hold. Moreover, by Lemma 3.3, we have $\int_{t_0}^{\infty} \widetilde{Q}(t)dt < \infty$, which is a contradiction. Hence, we can conclude that both $x(t)$ and $z(t)$ are bounded, Lemma 3.1 now implies that for $z(t)$ either Lemma 3.1 (jj) or (jjj) holds.

If the case Lemma 3.1 (jj) holds, then Lemma 3.2 ensures that $\lim_{t\to\infty} x(t) = 0$. On the other hand, if the case Lemma 3.1 (jjj) holds, then there exists a finite $\lim_{t\to\infty} z(t) = -b < 0$. We know that $x(t) > 0$ and $x(t)$ is bounded, so $\limsup_{t\to\infty} x(t) = a \geqslant 0$.

We next claim that $a = 0$. If not, then there exists a sequence t_k such that $\lim_{k\to\infty} t_k = \infty$ and $\lim_{k\to\infty} x(t_k) = a$. It is easy to see that for $\varepsilon = a(1-\mu)/(2\mu) > 0$, we have $x(t_k) < a + \varepsilon$ for k large enough, and

$$0 > -b = \lim_{k\to\infty} z(t_k) \geqslant \lim_{k\to\infty} (x(t_k) - \mu(a+\varepsilon)) = \frac{a}{2}(1-\mu) > 0.$$

This is a contradiction. Thus, $a = 0$, i.e., $\lim_{t\to\infty} x(t) = 0$. Hence, we complete the proof. □

THEOREM 3.2. *Assume that (2.1) holds. If*

$$\liminf_{t\to\infty} \frac{1}{\widetilde{A}_0(t)} \int_t^{\infty} \Big(\frac{\widetilde{A}_0^{\alpha+1}(s)}{r(s)}\Big)^{1/\alpha} ds > \frac{1}{(\alpha+1)^{(\alpha+1)/\alpha}}, \tag{3.5}$$

then the any solution $x(t)$ of Eq.(1.1) is oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.

The proof of this theorem is similar to that of Theorem 2.2 and can be omitted.

Denote

$$\widetilde{Q}_0(t) = \int_t^{\infty} \Big(\frac{\sigma^2(s)}{s}\Big)^{\alpha} q(s)ds,\ t \geqslant t_0.$$

Then, by Theorem 3.2, we have the following result.

COROLLARY 3.1. *Assume that (2.1) holds. If*

$$\liminf_{t\to\infty} \frac{1}{\widetilde{Q}_0(t)} \int_t^{\infty} \Big(\frac{\widetilde{Q}_0^{\alpha+1}(s)}{r(s)}\Big)^{1/\alpha} ds > \frac{2}{(\alpha+1)^{(\alpha+1)/\alpha}}, \tag{3.6}$$

then any solution $x(t)$ of Eq.(1.1) is oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.

By Lemma 3.4, similar to the proof of Theorem 2.3, we have

THEOREM 3.3. *Assume that (2.1) holds. If there is a positive integer m such that either of the following conditions hold.*

$$\int_{t_0}^{\infty} \widetilde{Q}(t) \exp\Big(\alpha \int_{t_0}^{t} \Big(\frac{\widetilde{A}_m(s)}{r(s)}\Big)^{1/\alpha} ds\Big) dt = \infty, \tag{3.7}$$

or

$$\int_{t_0}^{\infty} \Big(\frac{\widetilde{A}_m(t)}{r(t)}\Big)^{1/\alpha} \widetilde{A}_0(t) \exp\Big(\alpha \int_{t_0}^{t} \Big(\frac{\widetilde{A}_m(s)}{r(s)}\Big)^{1/\alpha} ds\Big) dt = \infty, \tag{3.8}$$

then any solution $x(t)$ of Eq.(1.1) is oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.

EXAMPLE 3.1. Consider the third order differential equation

$$\Big(t^2 |z''(t)| z''(t)\Big)' + \frac{c_2}{t^3} \Big|x\Big(\frac{t}{\sqrt{2}}\Big)\Big| x\Big(\frac{t}{\sqrt{2}}\Big) = 0, \quad t \geqslant 1, \tag{3.9}$$

where $z(t) = x(t) - \frac{1}{3}x(\frac{t}{2})$, $c_2 > 0$. For Eq.(1.1), let

$$\alpha = 2,\ r(t) = t^2,\ q(t) = \frac{c_2}{t^3},\ p(t) = -\frac{1}{3},\ \tau(t) = \frac{t}{\sqrt{2}} \quad \text{and} \quad \delta(t) = \frac{t}{2}.$$

Note that

$$\int_1^{\infty} \int_v^{\infty} \Big[\frac{1}{r(u)} \int_u^{\infty} q(s) ds\Big]^{1/2} du dv = \big(\frac{c_2}{2}\big)^{1/2} \int_1^{\infty} \frac{1}{v} dv = \infty,$$

i.e., (2.1) holds. For some $k_1, k_2 \in (0,1)$,

$$\widetilde{Q}(t) = \Big(\frac{k_1 k_2}{2}\Big)^2 \Big(\frac{\tau^2(t)}{t}\Big)^2 q(t) = \frac{c_2 (k_1 k_2)^2}{16} \frac{1}{t}.$$

Then

$$\int_1^{\infty} \widetilde{Q}(t) \exp\Big(2 \int_1^{t} \Big(\frac{\widetilde{A}_m(s)}{r(s)}\Big)^{1/2} ds\Big) dt \geqslant \frac{c_2 (k_1 k_2)^2}{16} \int_1^{\infty} \frac{1}{t} dt = \infty.$$

So, (3.7) holds. Therefore, by Theorem 3.3, every nonoscillatory solution of Eq.(3.9) converges to zero. In fact, let $c_2 = \frac{4}{9}(3 - 2^{\beta})^2 \beta^2 (\beta + 1)^3$, $\beta > 0$, one such solution is $x(t) = t^{-\beta}$.

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